Perfect Secrecy

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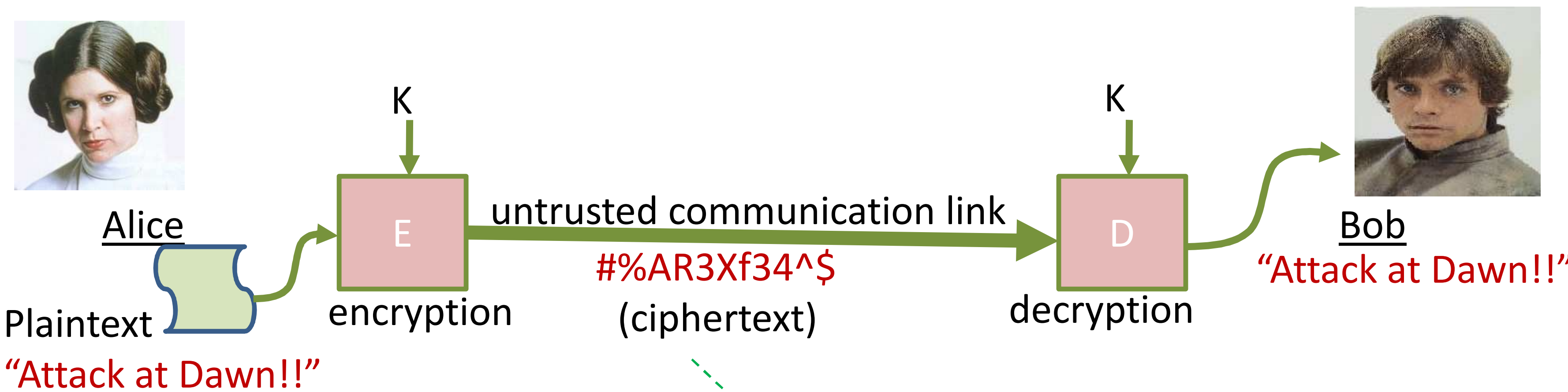
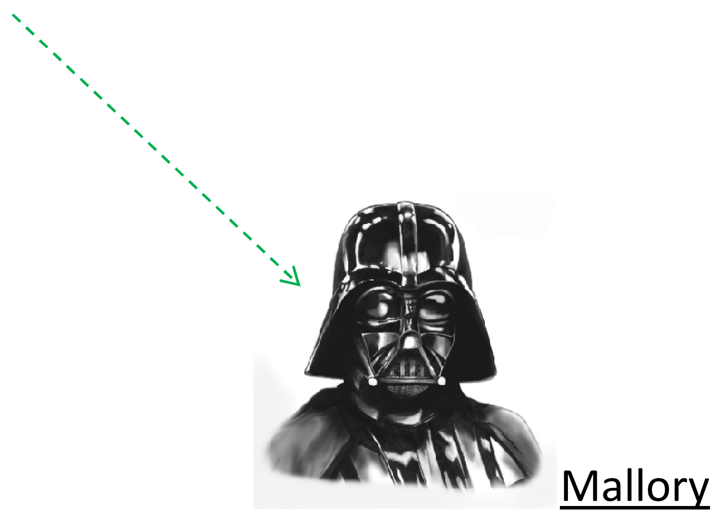
STINSON : chapter 2

Encryption





How do we design ciphers?

Cipher Models
(What are the goals of the design?)

Computation Security

My cipher can withstand all attacks with complexity less than 2^{2048}

The best attacker with the best computation resources would
take 3 centuries to attack my cipher

Provable Security (Hardness relative to a tough problem)

If my cipher can be broken then large numbers can be factored easily

Unconditional Security

My cipher is secure against all attacks irrespective of the
attacker's power.
I can prove this!!

This model is also known as **Perfect Secrecy.**
Can such a cryptosystem be built?
We shall investigate this.

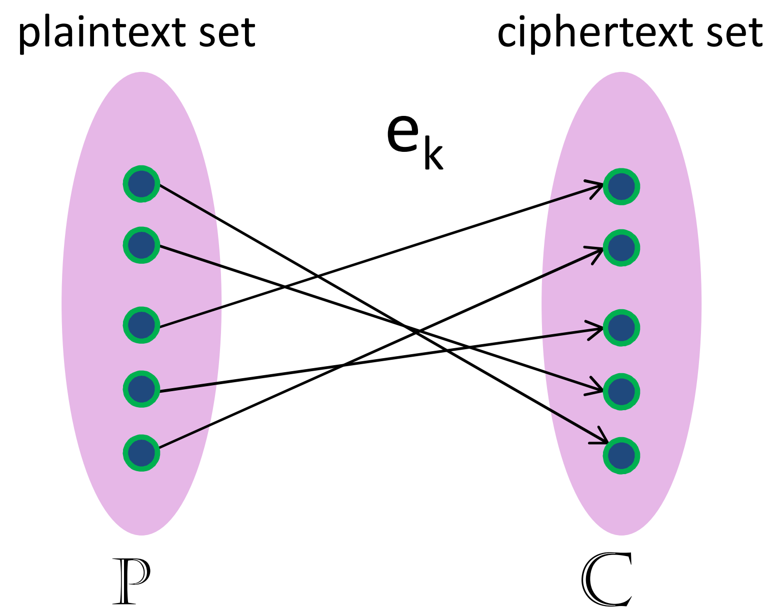
Analyzing Unconditional Security

- Assumptions
  - Ciphertext only attack model

    The attacker only has information about the ciphertext. The key and plaintext are secret.

- We first analyze a single encryption then relax this assumption by analyzing multiple encryptions with the same key

Encryption



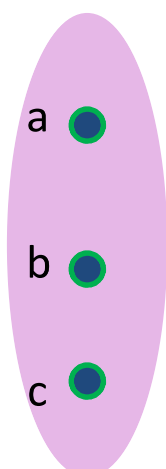
- For a given key, the encryption (e_k) defines an injective mapping between the plaintext set ($\mathbb{P}$) and ciphertext set ($\mathbb{C}$)
- We assume that the key and plaintext are independent

- Alice picks a plaintext $x \in \mathcal{P}$ and encrypts it to obtain a ciphertext $y \in \mathcal{C}$

Plaintext Distribution

Plaintext Distribution

- Let **X** be a discrete random variable over the set $\mathbb{P}$
- Alice chooses x from $\mathbb{P}$ based on some probability distribution
  - Let *Pr[**X** = x]* be the probability that x is chosen
  - This probability may depend on the language



$\mathbb{P}$

Plaintext set
Pr[**X**=a] = 1/2
Pr[**X**=b] = 1/3
Pr[**X**=c] = 1/6

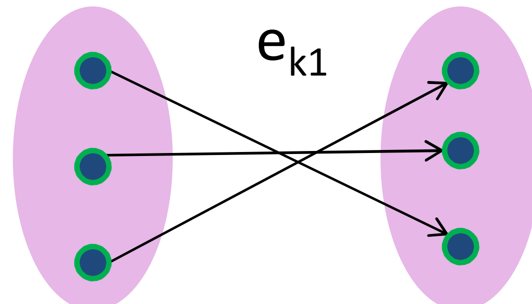
Note : Pr[a] + Pr[b] + Pr[c] = 1

Key Distribution

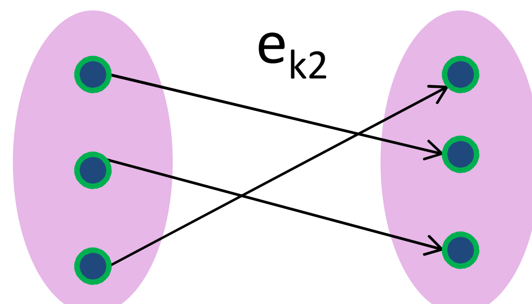
Key Distribution

- Alice & Bob agree upon a key k chosen from a key set $\mathbb{K}$
- Let ***K*** be a random variable denoting this choice

keyspace
Pr[**K**=k_1] = ¾
Pr[**K**=k_2] = ¼



There are two keys in the keyset
thus there are two possible encryption mappings



Ciphertext Distribution

- Let **Y** be a discrete random variable over the set $\mathbb{C}$
- The probability of obtaining a particular ciphertext y depends on the plaintext and key probabilities

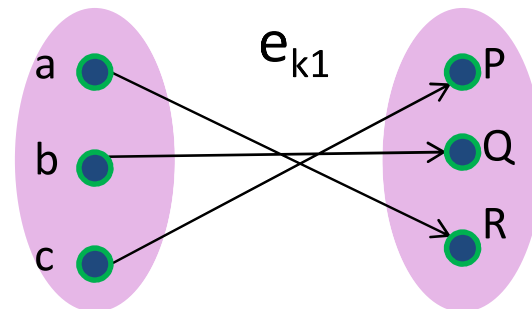
$$\Pr[Y = y] = \sum_{k} \Pr(k)\Pr(d_k(y))$$

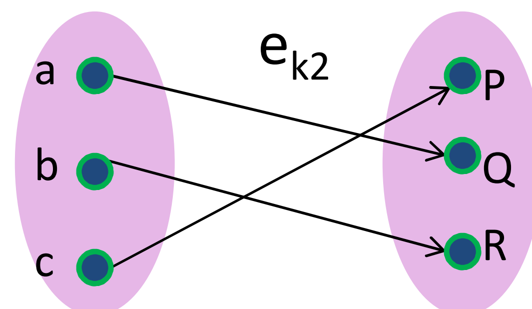
Pr[Y = P] = Pr(k_1) * Pr(c) + Pr(k_2) * Pr(c)
= (3/4 * 1/6) + (1/4 * 1/6) = **1/6**

Pr[Y = Q] = Pr(k_1) * Pr(b) + Pr(k_2) * Pr(a)
= (3/4 * 1/3) + (1/4 * 1/2) = **3/8**

Pr[Y = R] = Pr(k_1) * Pr(a) + Pr(k_2) * Pr(b)
= (3/4 * 1/2) + (1/4 * 1/3) = **11/24**

Note: Pr[Y=P] + Pr[Y=Q] + Pr[Y=R] = 1





plaintext
Pr[**X**=a] = 1/2
Pr[**X**=b] = 1/3
Pr[**X**=c] = 1/6

keyspace
Pr[**K**=k_1] = ¾
Pr[**K**=k_2] = ¼

Attacker's Probabilities

- The attacker wants to determine the plaintext ***x***
- Two scenarios
  - Attacker does not have y (a priori Probability)
    - Probability of determining ***x*** is simply ***Pr[x]***
    - Depends on plaintext distribution (eg. Language charcteristics)
  - Attacker has y (a posteriori probability)
    - Probability of determining x is simply ***Pr[x|y]***

A posteriori Probabilities

- How to compute the attacker's a posteriori probabilities? $\Pr[X = x \mid Y = y]$
  - Bayes' Theorem

$$\boxed{\Pr[x \mid y] = \frac{\Pr[x] \times \Pr[y \mid x]}{\Pr[y]}}$$

probability of the plaintext

probability of this ciphertext

?

The probability that y is obtained given x depends on the keys which provide such a mapping

$$\Pr[y \mid x] = \sum_{\{k \,:\, d_k(y) = x\}} \Pr[k]$$

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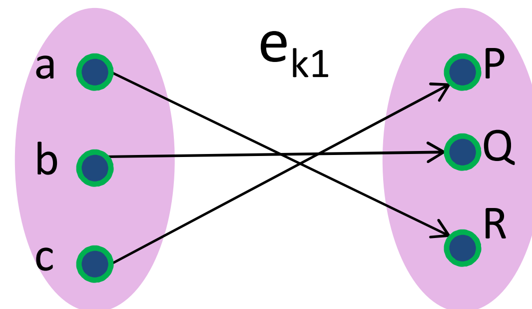
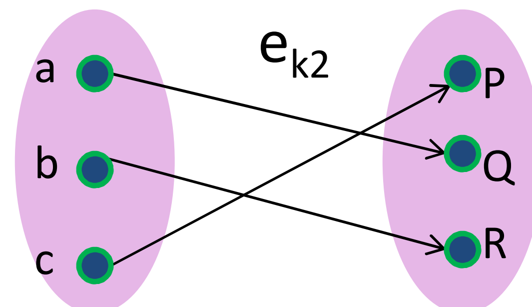
$Pr[y|x]$

$Pr[P|a] = 0$
$Pr[P|b] = 0$
$Pr[P|c] = 1$

$Pr[Q|a] = Pr[k_2] = 1/4$
$Pr[Q|b] = Pr[k_1] = 3/4$
$Pr[Q|c] = 0$

$Pr[R|a] = Pr[k_1] = 3/4$
$Pr[R|b] = Pr[k_2] = 1/4$
$Pr[R|c] = 0$





keyspace
$Pr[\mathbf{K}=k_1] = 3/4$
$Pr[\mathbf{K}=k_2] = 1/4$

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Computing A Posteriori Probabilities

$$\Pr[x \mid y] = \frac{\Pr[x] \times \Pr[y \mid x]}{\Pr[y]}$$

plaintext	ciphertext	Pr[y|x]
Pr[**X**=a] = 1/2	Pr[**Y**=P] = 1/6	Pr[P|a] = 0 Pr[P|b] = 0 Pr[P|c] = 1
Pr[**X**=b] = 1/3	Pr[**Y**=Q] = 3/8	Pr[Q|a] = ¼ Pr[Q|b] = ¾ Pr[Q|c] = 0
Pr[**X**=c] = 1/6	Pr[**Y**=R] = 11/24	Pr[R|a] = ¾ Pr[R|b] = ¼ Pr[R|c] = 0

Pr[a|P] = 0	Pr[b|P] = 0	Pr[c|P] = 1
Pr[a|Q] = 1/3	Pr[b|Q] = 2/3	Pr[c|Q] = 0
Pr[a|R] = 9/11	Pr[b|R] = 2/11	Pr[c|R] = 0

If the attacker sees ciphertext ***P*** then she would know the plaintext was ***c***
If the attacker sees ciphertext ***R*** then she would know ***a*** is the most likely plaintext
Not a good encryption mechanism!!

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Perfect Secrecy

- Perfect secrecy achieved when

a posteriori probabilities = a priori probabilities

$$\boxed{\Pr[x \mid y] = \Pr[x]}$$

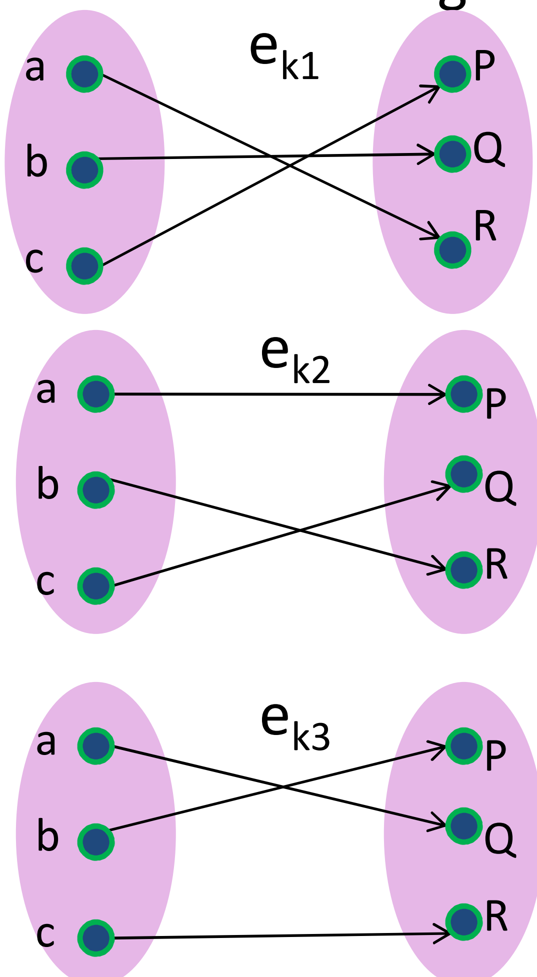
i.e the attacker learns nothing from the ciphertext

Perfect Secrecy Example

- Find the a posteriori probabilities for the following scheme
- Verify that it is perfectly secret.

plaintext
Pr[**X**=a] = 1/2
Pr[**X**=b] = 1/3
Pr[**X**=c] = 1/6

keyspace
Pr[**K**=k_1] = 1/3
Pr[**K**=k_2] = 1/3
Pr[K=k_3] = 1/3



Observations on Perfect Secrecy

Perfect Secrecy iff

Follows from
Baye's theorem

$$\boxed{\Pr[Y = y \mid X = x] = \Pr[Y = y]}$$

Perfect Indistinguishability

$\forall x_1, x_2 \in P$

$$\boxed{\Pr[Y = y \mid X = x_1] = \Pr[Y = y \mid X = x_2]}$$

Perfect secrecy has nothing to do with plaintext distribution.
Thus a crypto-scheme will achieve perfect secrecy irrespective of the language used in the plaintext.

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Shift Cipher with a Twist

- Plaintext set : $\mathbb{P}$ = {0,1,2,3 …, 25}
- Ciphertext set : $\mathbb{C}$ = {0,1,2,3 …, 25}
- Keyspace : $\mathbb{K}$ = {0,1,2,3 …, 25}
- Encryption Rule : $e_K(x) = (x + K) \bmod 26,$
- Decryption Rule : $d_k(x) = (x - K) \bmod 26$

  where $K \in \mathbb{K}$ and $x \in \mathbb{P}$

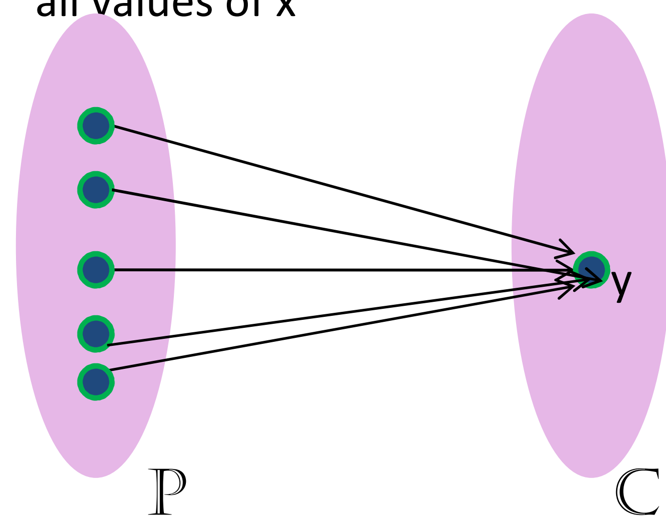
The Twist : the key changes after every encryption

The Twisted Shift Cipher is Perfectly Secure

$$
\begin{aligned}
\mathbf{Pr}[\mathbf{y} = y] &= \sum_{K \in \mathbb{Z}_{26}} \mathbf{Pr}[\mathbf{K} = K]\mathbf{Pr}[\mathbf{x} = d_K(y)] \\
&= \sum_{K \in \mathbb{Z}_{26}} \frac{1}{26}\mathbf{Pr}[\mathbf{x} = y - K] \\
&= \frac{1}{26} \sum_{K \in \mathbb{Z}_{26}} \mathbf{Pr}[\mathbf{x} = y - K]. \\
&= \frac{1}{26}
\end{aligned}
$$

Keys chosen with uniform probability

This is 1 because the sum is over all values of x



$$
\begin{aligned}
\mathbf{Pr}[y|x] &= \mathbf{Pr}[\mathbf{K} = (y - x) \bmod 26] \\
&= \frac{1}{26}
\end{aligned}
$$

For every pair of y and x, there is exactly one key . Probability of that key is 1/26

The Twisted Shift Cipher is Perfectly Secure

$$\begin{aligned}\mathbf{Pr}[\mathbf{y} = y] &= \sum_{K \in \mathbb{Z}_{26}} \mathbf{Pr}[\mathbf{K} = K]\mathbf{Pr}[\mathbf{x} = d_K(y)] \\ &= \sum_{K \in \mathbb{Z}_{26}} \frac{1}{26}\mathbf{Pr}[\mathbf{x} = y - K] \\ &= \frac{1}{26} \sum_{K \in \mathbb{Z}_{26}} \mathbf{Pr}[\mathbf{x} = y - K]. \\ &= \frac{1}{26}\end{aligned}$$

$$\boxed{\begin{aligned}\mathbf{Pr}[x|y] &= \frac{\mathbf{Pr}[x]\mathbf{Pr}[y|x]}{\mathbf{Pr}[y]} \\ &= \frac{\mathbf{Pr}[x]\frac{1}{26}}{\frac{1}{26}} \\ &= \mathbf{Pr}[x],\end{aligned}}$$

$$\begin{aligned}\mathbf{Pr}[y|x] &= \mathbf{Pr}[\mathbf{K} = (y - x) \bmod 26] \\ &= \frac{1}{26}\end{aligned}$$

Shannon's Theorem

If $|\mathbb{K}| = |\mathbb{C}| = |\mathbb{P}|$ then the system provides perfect secrecy iff

(1) every key is used with equal probability $1/|\mathbb{K}|$, and

(2) for every x $\in \mathbb{P}$ and y $\in \mathbb{C}$, there exists a unique key k $\in \mathbb{K}$ such that $e_k(x) = y$

Intuition :

Every y $\in \mathbb{C}$ can result from any of the possible plaintexts x

Since |K| = |P| there is exactly one mapping from each plaintext to y

Since each key is equi-probable, each of these mappings is equally probable

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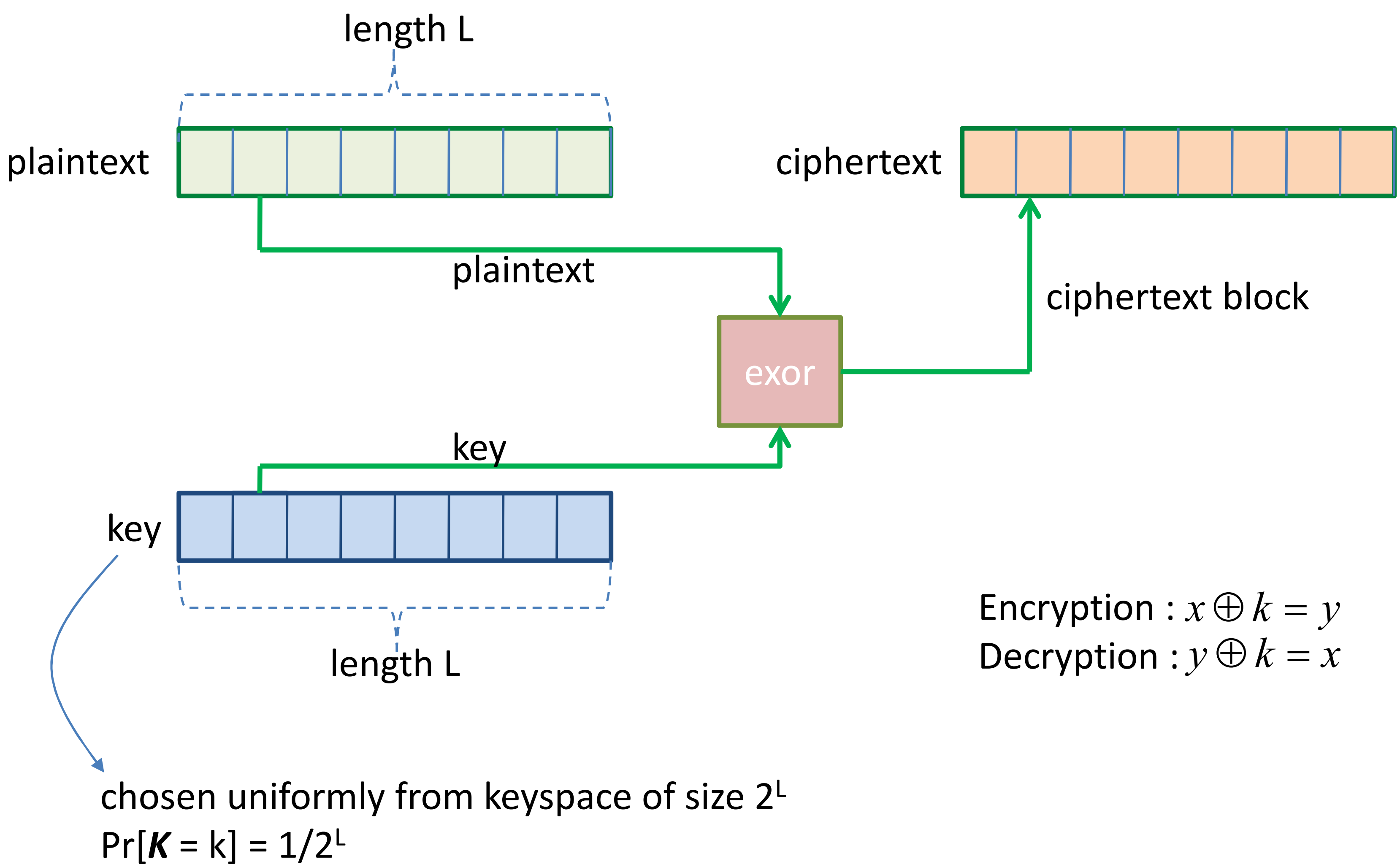
One Time Pad (Verman's Cipher)



Encryption : $x \oplus k = y$

Decryption : $y \oplus k = x$

One Tme Pad (Example)

e=000 h=001 i=010 k=011 l=100 r=101 s=110 t=111

Encryption: Plaintext ⊕ Key = Ciphertext

	h	e	i	l	h	i	t	l	e	r
Plaintext:	001	000	010	100	001	010	111	100	000	101
Key:	111	101	110	101	111	100	000	101	110	000
Ciphertext:	110	101	100	001	110	110	111	001	110	101
	s	r	l	h	s	s	t	h	s	r

One Time Pad is Perfectly Secure

- Proof using indistinguishability

$$\Pr[Y = y \mid X = x] = \Pr[X = x, K = k \mid X = x] \qquad \text{from } x \oplus k = y$$
$$= \Pr[K = k] = \frac{1}{2^L}$$

$$\Pr[Y = y \mid X = x_1] = \frac{1}{2^L} = \Pr[Y = y \mid X = x_2]$$
$$\forall x_1, x_2 \in X$$

This implies perfect Indistinguishability
that is independent of the plaintext distribution

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Limitations of Perfect Secrecy

- Key must be at least as long as the message
  - Limits applicability if messages are long

- Key must be changed for every encryption
  - If the same key is used twice, then an adversary can compute the ex-or of the messages

$$x_1 \oplus k = y_1$$
$$x_2 \oplus k = y_2$$
$$x_1 \oplus x_2 = y_1 \oplus y_2$$

The attacker can then do language analysis to determine y_1 and y_2

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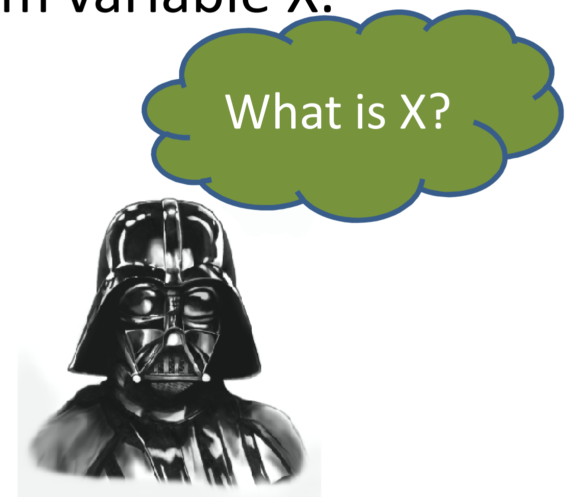
Computational Security

- Perfect secrecy is difficult to achieve in practice
- Instead we use a crypto-scheme that cannot be ***broken in reasonable time*** with ***reasonable success***
- This means,
  - Security is only achieved against adversaries that run in polynomial time
  - Attackers can potentially succeed with a very small probability (attackers need to be very lucky to succeed)

Quantifying Information

Quantifying Information

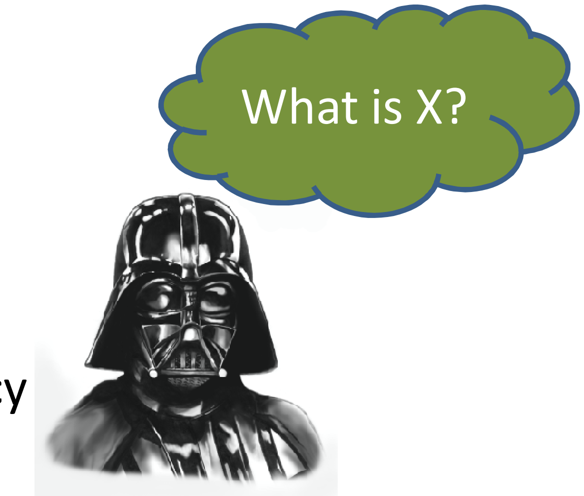
- Alice thinks of a number (0 or 1)
- The choice is denoted by a discrete random variable X.

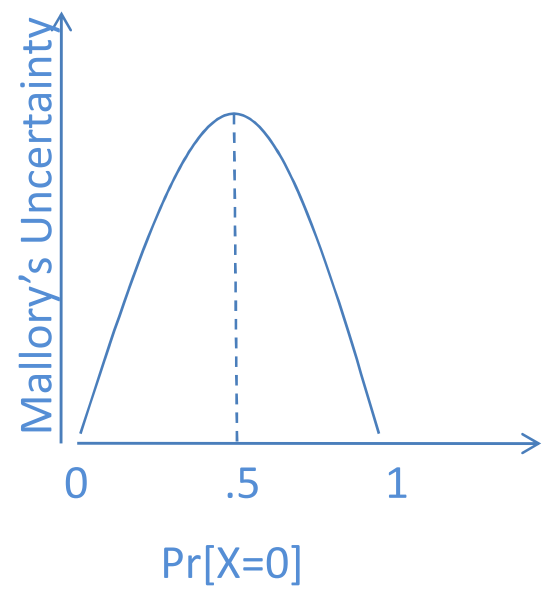


- What is the information in X?
- What is Mallory’s uncertainty about X?
  - Depends on the probability distribution of X

Uncertainty

- Lets assume Mallory know this probability distribution.
- If Pr[X = 1] = 1 and Pr[X = 0] = 0
  - Then Mallory can determine with 100% accuracy
- If Pr[X = 0] = .75 and Pr[X = 1] = .25
  - Mallory will guess X as 0, and gets it right 75% of the time
- If Pr[X=0] = Pr[X = 1] = 0.5
  - Mallory's guess would be similar to a uniformly random guess. Gets it right ½ the time.





Entropy (Quantifying Information)

- Suppose we consider a discrete R.V. X taking values from the set $\{x_1, x_2, x_3, \ldots, x_n\}$,
  each symbol occurring with probability
  $\{p_1, p_2, p_3, \ldots, p_n\}$
- Entropy is defined as the minimum number of bits (on average) that is required to represent a string from this set?

Probability that the ith symbol occurs

$$\underbrace{H(X)}_{\text{Entropy of X}} = \sum_{i=1}^{n} p_i \underbrace{\log_2\left(\frac{1}{p_i}\right)}_{\text{Bits to encode the ith symbol}}$$

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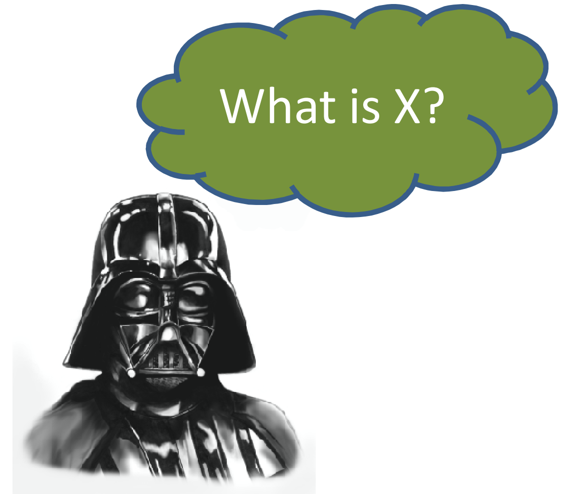
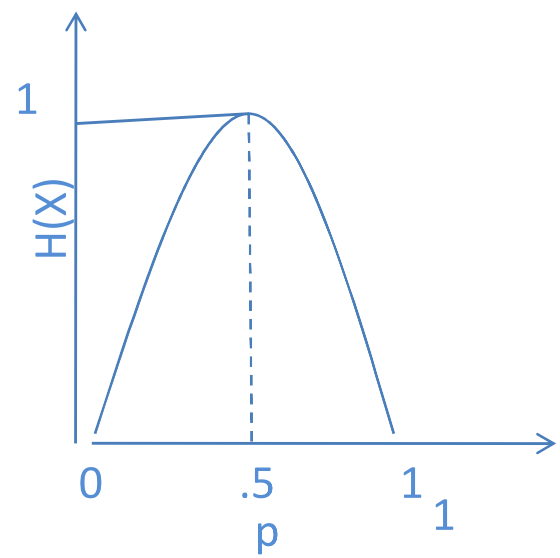
What is the Entropy of X?



$Pr[X=0] = p$ and $Pr[X=1] = 1 - p$

$H(X) = -\,p\log_2 p - (1-p)\log_2(1-p)$

$H(X)_{p=0} = 0,\ H(X)_{p=1} = 0,\ H(X)_{p=.5} = 1$



using $\lim_{p \to 0} (p \log p) = 0$

Properties of H(X)

- If X is a random variable, which takes on values {1,2,3,….n} with probabilities p_1, p_2, p_3, ….p_n, then

1. $H(X) \le \log_2 n$

2. When $p_1= p_2=p_3= \ldots p_n = 1/n$ then $H(X) = \log_2 n$

   Example an 8 face dice.
   If the dice is fair, then we obtain the maximum entropy of 3 bits
   If the dice is unfair, then the entropy is < 3 bits

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Entropy and Coding

- Entropy quantifies Information content

  "Can we encode a message M in such a way that the average length is as short as possible and hopefully equal to H(M)?"

  Huffman Codes :

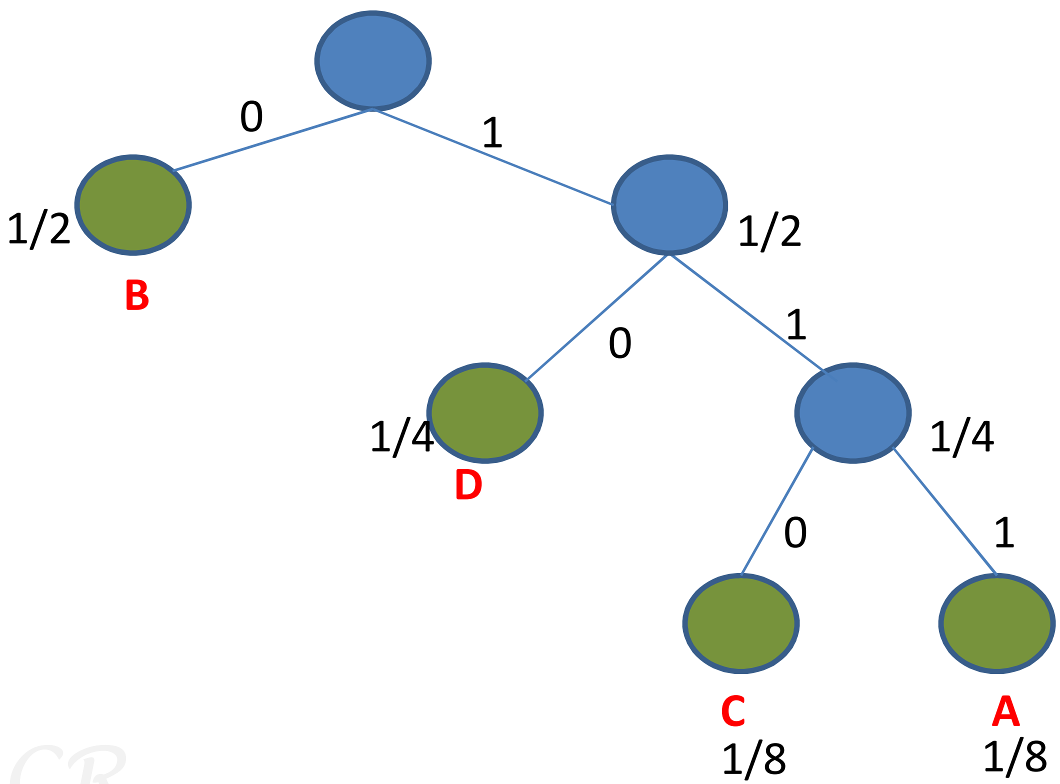
  allocate more bits to least probable events
  allocate less bits to popular events

Example

- S = {A, B, C, D} are 4 symbols
- Probability of Occurrence is :

  P(A) = 1/8, P(B) = ½, P(C) = 1/8, P(D) = 1/4

Encoding
A : 111
B : 0
C : 110
D: 10



To decode, with each bit traverse the tree from root until you reach a leaf.

Decode this?
1101010111

Example : Average Length and Entropy

- S = {A, B, C, D} are 4 symbols
- Probability of Occurrence is :

  p(A) = 1/8, p(B) = ½, p(C) = 1/8, p(D) = ¼

Encoding
A : 111
B : 0
C : 110
D: 10

- Average Length of Huffman code :

  3*p(A) + 1*p(B) + 3*p(C) + 2*p(D) = 1.75

- Entropy H(S) =

  $-1/8 \log_2(8) - ½ \log_2(2) - 1/8 \log_2(8) - ¼ \log_2(4)$

  = 1.75

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Measuring the Redundancy in a Language

- Let S be letter in a language (eg. S = {A,B,C,D})
- $\mathbf{S} = S \times S \times S \times S \times S \times S \, (k \text{ times})$ is a set representing messages of length k
- Let $S^{(k)}$ be a random variable in $\mathbb{S}$
- The average information in each letter is given by the rate of $S^{(k)}$.

$$r_k = \frac{H(S^{(k)})}{k}$$

- r_k for English is between 1.0 and 1.5 bits/letter

Measuring the Redundancy in a Language

- Absolute Rate : The maximum amount of information per character in a language
  - the absolute rate of language S is $R = \log_2 |S|$
  - For English, $|S| = 26$, therefore $R = 4.7$ bits / letter

- Redundancy of a language is

  $$D = R - r_k$$

  - For English when rk = 1, then D = 3.7 → around 79% redundant

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Example (One letter analysis)

- Consider a language with 26 letters of the set S = {s_1, s_2, s_3, ….., s_{26}}. Suppose the language is characterized by the following probabilities. **What is the language redundancy?**

$$P(s_1) = \frac{1}{2},\ P(s_2) = \frac{1}{4}$$

$$P(s_i) = \frac{1}{64} \quad for \quad i = 3,4,5,6,7,8,9,10$$

$$P(s_i) = \frac{1}{128} \quad for \quad i = 11,12,...,26$$

Absolute Rate

$$R = \log 26 = 4.7$$

Rate of the Language for 1 letter analysis

$$\begin{aligned} r_1 &= H(S^{(1)}) \\ &= \sum_{i=1}^{26} P(s_i) \log \frac{1}{P(s_i)} \\ &= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + 8\left(\frac{1}{64}\log 64\right) + 16\left(\frac{1}{128}\log 128\right) \\ &= \frac{1}{2} + \frac{1}{2} + \frac{6}{8} + \frac{7}{8} = 2.625 \end{aligned}$$

Language Redundancy

$$D = R - r_1 = 4.7 - 2.625 = 2.075$$

Language is ~70% redundant

Example (Two letter analysis)

- In the set S = {s_1, s_2, s_3, ….., s_{26}}, suppose the diagram probabilites is as below. **What is the language redundancy?**

$$P(s_{i+1} \mid s_i) = P(s_{i+2} \mid si) = \frac{1}{2} \quad for \quad i = 1 \ to \ 24$$

$$P(s_{26} \mid s_{25}) = P(s_1 \mid s_{25}) = P(s_1 \mid s_{26}) = P(s_2 \mid s_{26}) = \frac{1}{2}$$

$$all \ \ other \ \ probabilities \ \ are \ \ 0$$

$$P(s_1, s_2) = P(s_2 \mid s_1) \times P(s_1) = 1/4; \ P(s_1, s_3) = P(s_3 \mid s_1) \times P(s_1) = 1/4$$
$$P(s_2, s_3) = P(s_3 \mid s_2) \times P(s_2) = 1/8; \ P(s_2, s_4) = P(s_4 \mid s_2) \times P(s_2) = 1/8$$
$$P(s_i, s_{i+1}) = P(s_{i+1} \mid s_i) P(s_i) = 1/128 \quad for \ \ i = 3, 4, \ ......, 10$$
$$P(s_i, s_{i+2}) = P(s_{i+2} \mid s_i) P(s_i) = 1/128 \quad for \ \ i = 3, 4, \ ......, 10$$
$$P(s_i, s_{i+1}) = P(s_{i+1} \mid s_i) P(s_i) = 1/256 \quad for \ \ i = 11, 12, \ ......, 24$$
$$P(s_i, s_{i+2}) = P(s_{i+2} \mid s_i) P(s_i) = 1/256 \quad for \ \ i = 11, 12, \ ......, 24$$
$$P(s_{25}, s_{26}) = P(s_{25}, s_1) = P(s_{26}, s_1) = P(s_{26}, s_2) = 1/256$$

Rate of the Language for 2 letter analysis

$$\begin{aligned} r_2 &= H(S^{(2)})/2 \\ &= \frac{1}{2} \sum_{i,j=1}^{26} P(s_i, s_j) \log \frac{1}{P(s_i, s_j)} \\ &= \frac{1}{2}\left[2\left(\frac{1}{4}\log 4\right) + 2\left(\frac{1}{8}\log 8\right) + 16\left(\frac{1}{128}\log 128\right) + 32\left(\frac{1}{256}\log 256\right) \right] \\ &= \frac{1}{2}\left[1 + \frac{3}{4} + \frac{7}{8} + 1 \right] = \frac{3.625}{2} = 1.8125 \end{aligned}$$

Language Redundancy

$$D = R - r_2 = 4.7 - 1.8125 = 2.9$$

Language is ~60% redundant

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Observations

$$\textit{Single letter analysis}\ : r_1 = H(S^{(1)}) = 2.625;\ \ D = 2.075$$

$$\textit{Two letter analysis}\ : H(S^{(2)}) = 3.625;\ \ r_2 = 1.8125;\ \ D = 2.9$$

- H(S(2)) – H(S(1)) = 1 bit
  - why?
- As we increase the message size
  - Rate reduces; inferring less information per letter
  - Redundancy increases

Conditional Entropy

- Suppose X and Y are two discrete random variables, then conditional entropy is defined as

$$H(X|Y) = \sum_y p(y) \sum_x p(x|y) \log_2 \left(\frac{1}{p(x|y)} \right)$$

$$= \sum_x \sum_y p(y).p(x|y) \log_2 \left(\frac{p(x)}{p(x,y)} \right)$$

- Conditional entropy means ….
  - What is the remaining uncertainty about X given Y
  - H(X|Y) ≤ H(X) with equality when X and Y are independent

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Derive using the fact that p(a|b) = p(a,b) / p(b)

Joint Entropy

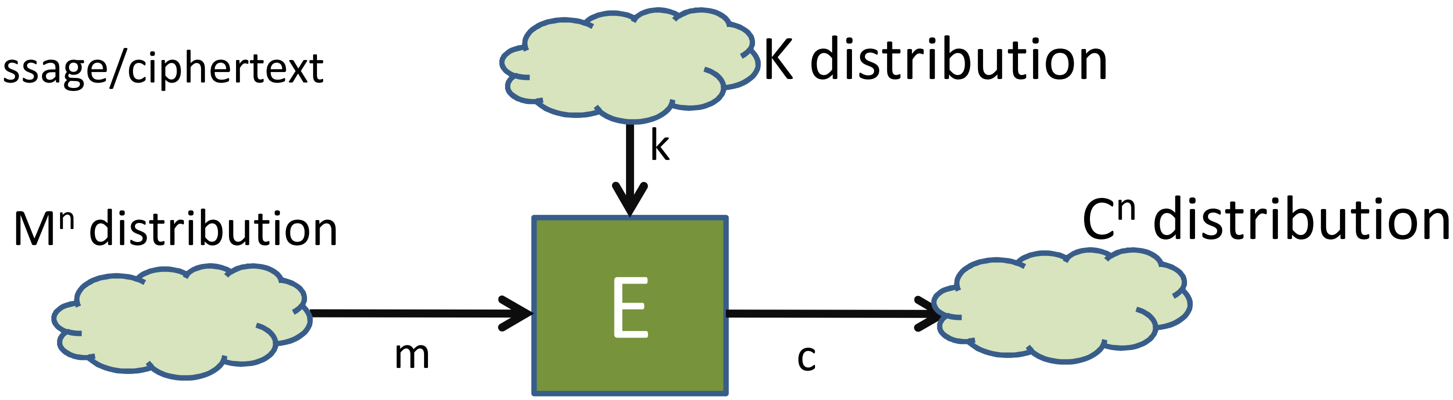
- Suppose X and Y are two discrete random variables, and p(x,y) the value of the joint probability distribution when X=x and Y=y
- Then the joint entropy is given by

$$H(X,Y) = \sum_{y} \sum_{x} p(x,y) \log_2 \left(\frac{1}{p(x,y)} \right)$$

- The joint entropy is the average uncertainty of 2 random variables

Entropy and Encryption

n: length of message/ciphertext



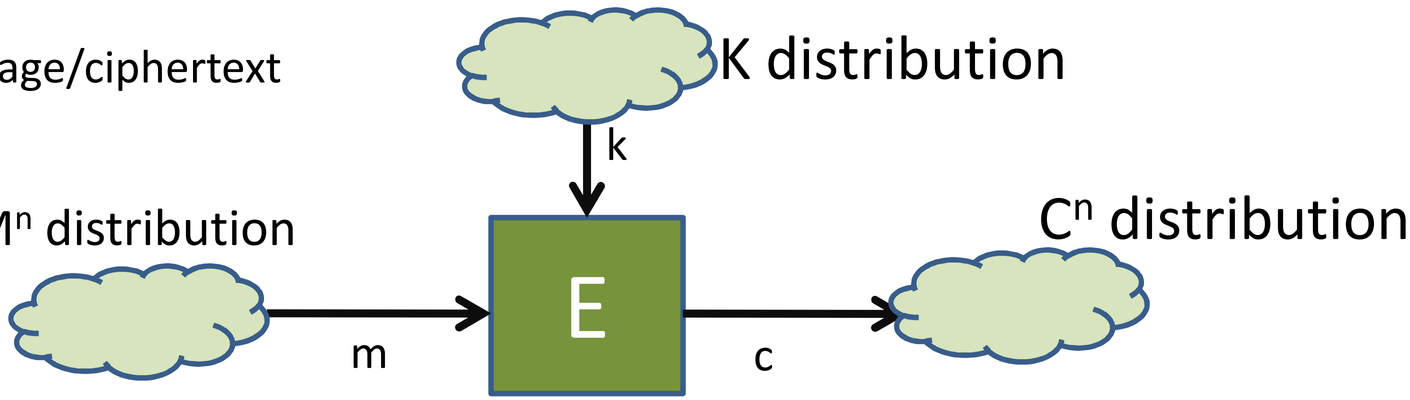
- There are three entropies: $H(P^{(n)})$, $H(K)$, $H(C^{(n)})$
- Message Equivocation :

  If the attacker can view n ciphertexts, what is his uncertainty about the message

$$H(M^{(n)} \mid C^{(n)}) = \sum_{c \in C^n} p(c) \sum_{m \in M^n} p(m \mid c) \log_2 \left(\frac{1}{p(m \mid c)} \right)$$

Entropy and Encryption

n: length of message/ciphertext



- Key Equivocation :

  If the attacker can view n ciphertexts, what is his uncertainty about the key

$$H(K \mid C^{(n)}) = \sum_{c \in C^n} p(c) \sum_{m \in M^n} p(k \mid c) \log_2 \left(\frac{1}{p(k \mid c)} \right)$$

Unicity Distance

$$H(K \mid C^{(n)}) = \sum_{c \in C^n} p(c) \sum_{m \in M^n} p(k \mid c) \log_2 \left(\frac{1}{p(k \mid c)} \right)$$

- As n increases, $H(K|C^{(n)})$ reduces…
  – This means that the uncertainty of the key reduces as the attacker observes more ciphertexts
- Unicity distance is the value of n for which $H(K \mid C^{(n)}) \approx 0$
  – This means, the entire key can be determined in this case

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Unicity Distance and Classical Ciphers

Cipher	Unicity Distance (for English)
Caesar’s Cipher	1.5 letters
Affine Cipher	2.6 letters
Simple Substitution Cipher	27.6 letters
Permutation Cipher	0.12 (block size = 3) 0.66 (block size = 4) 1.32 (block size = 5) 2.05 (block size = 6)
Vigenere Cipher	1.47d (d is the key length)

Product Ciphers

- Consider a cryptosystem where $\mathbb{P}=\mathbb{C}$ (this is an endomorphic system)
  - Thus the ciphertext and the plaintext set is the same
- Combine two ciphering schemes to build a product cipher

Given two endomorphic crypto-systems

$$S_1 : x = d_{K_1}(e_{K_1}(x))$$

$$S_2 : x = d_{K_2}(e_{K_2}(x))$$

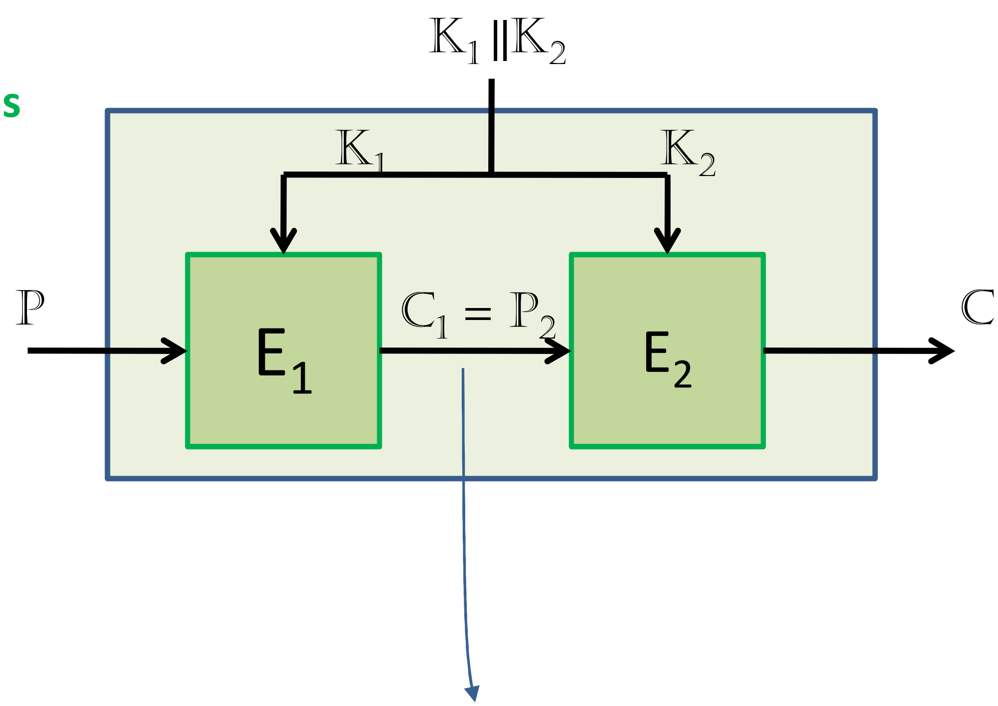
Resultant Product Cipher

$$S_1 \times S_2$$

$$e_{(K_1,K_2)}(x) = e_{K_2}(e_{K_1}(x))$$

$$d_{(K_1,K_2)}(x) = d_{K_2}(d_{K_1}(x))$$

Resultant Key Space $K_1 \times K_2$



Ciphertext of first cipher fed as input to the second cipher

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Product Ciphers

- Consider a cryptosystem where $\mathbb{P}=\mathbb{C}$ (this is an endomorphic system)
  - Thus the ciphertext and the plaintext set is the same
- Combine two ciphering schemes to build a product cipher

Given two endomorphic crypto-systems

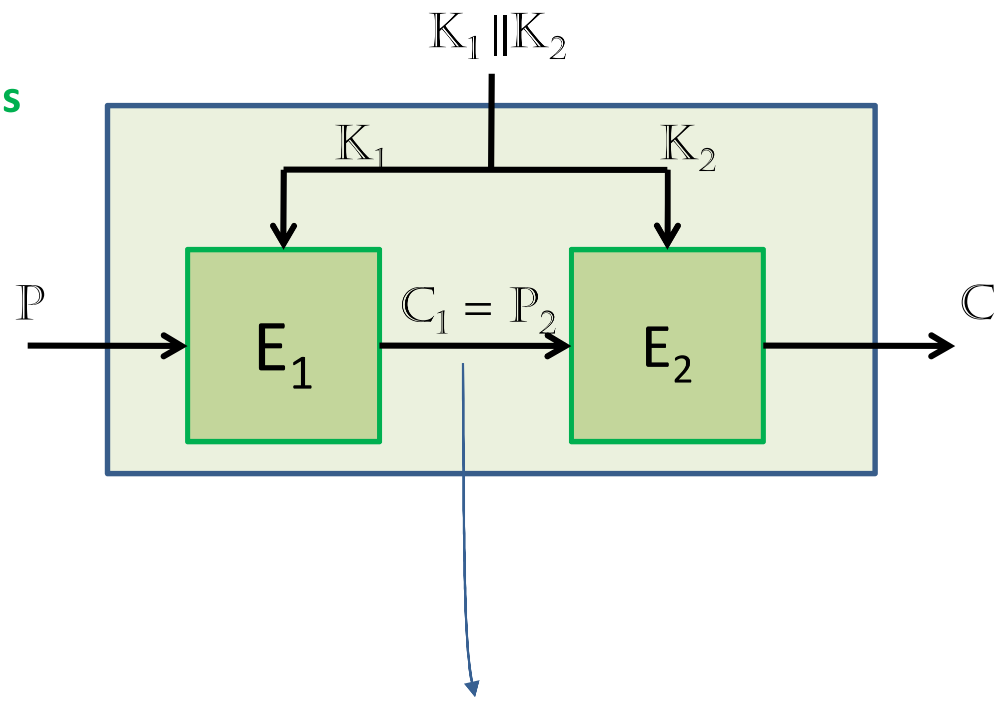
$$S_1 : (P, P, K_1, E_1, D_1)$$

$$S_2 : (P, P, K_2, E_2, D_2)$$

Resultant Product Cipher

$$S_1 \times S_2 : (P, P, K_1 \times K_2, E, D)$$

Resultant Key Space $K_1 \times K_2$



Ciphertext of first cipher fed as input to the second cipher

Affine Cipher is a Product Cipher

Multiplicative Cipher Shift Cipher

- $\mathbb{P} = \mathbb{C} = \{0, 1, 2, \ldots 25\}$

Affine Cipher = M x S

Encryption ($e_a(x)$) : $y = ax \bmod 26$
Decryption ($d_a(x)$) : $x = a^{-1}y \bmod 26$

Encryption ($e_b(x)$) : $y = x+b \bmod 26$
Decryption ($d_b(x)$) : $x = y-b \bmod 26$

- Affine cipher : $y = ax + b \bmod 26$
- Size of Key space is
  - Size of key space for Multiplicative cipher * Size of keyspace for shift cipher
  - $12 * 26 = 312$

Is S x M same as the Affine Cipher

- S x M : $y = a(x + b) \bmod 26$
  $= ax + ba \bmod 26$
- Key is (b,a)
- ba mod 26 is some b' such that
  $a^{-1}b' = b \bmod 26$
- This can be represented as an Affine cipher,
  $$y = ax + b' \bmod 26$$

Thus affine ciphers are commutable (i.e. S x M = M x S)

Create a non-commutable product ciphers

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Idempotent Ciphers

- If $S_1 : (P, P, K, E_1, D_1)$ is an endomorphic cipher
- then it is possible to construct product ciphers of the form S_1 x S_1, denoted $S^2 \; : (P, P, K \times K, E, D)$
- If $S^2 = S$ then the cipher is called idempotent cipher

Show that the simple substitution cipher is idempotent

Does the security of the newly formed cipher increase?

In a non-idempotent cipher, however the security may increase.

Iterative Cipher

- An n-fold product of this is **S x S x S … (n times) = S^n** is an iterative cipher

All modern block ciphers like DES, 3-DES, AES, etc. are iterative, non-idempotent, product ciphers.

We will see more about these ciphers next!!

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