Depth Lower Bounds and Non-monotonicity

(for Circuits with Sparse Orientation)

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Overview

- Known Lower Bounds
- Orientation
- Depth Lower Bounds vs Weight
- Barriers and Structure
- Proof Sketch
- Open Problems

Quick Recap

- $f : \{0,1\}^n \to \{0,1\}$.
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  - Depth $\Omega(\log n)$ for any function that depends on all input variables.
  - $\Omega(s(n))$ size $\implies$ $\Omega(\log s(n))$ depth.
- Known circuit lower bounds for general circuits are depressingly weak ($3.011n$ on size and $(3-\epsilon)\log n$ on depth).

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- Non-monotonicity does help in reducing circuit depth !!.
- (2014 Göös-Pitassi) Function in Monotone **NP** requiring $\Omega(n/\log n)$ monotone depth.

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- (2004 Jukna) Multi-output function such that $n^{\log n}$ size is required even when $\log n - O(\log \log n)$ negations are allowed.

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- (2015 Rossman) A function using s-t connectivity cannot be in $\mathbf{NC}^1$ using only $(\frac{1}{2} - \epsilon) \log n$ negations.

Against Non-monotone Circuits : Depth

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This work :

- Restriction (high-level idea) : Circuits where every internal gate computes a function which is not "far" from monotone.
- Main Result (high-level view) : A trade-off between "far"-ness and circuit depth lower bound.

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Orientation of Boolean Functions

- Let C be a DeMorgan circuit computing f with minimum number of negations.
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Orientation of a function $f : \{0,1\}^n \to \{0,1\}$ is a $\beta \in \{0,1\}^n$ such that there is a monotone function $h : \{0,1\}^{2n} \to \{0,1\}$ with $\forall x, f(x) = h(x, x \oplus \beta)$.

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Property : For a function f the minimal orientation is unique.

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Weight n orientation is sufficient to compute any function.

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- Target : $O(n \log n)$ densely oriented gates.
- Can we handle higher weight β's if we restrict the number of non-trivially oriented gates?
- **No**. There exists a (non-explicit) monotone function f which cannot be computed by $\omega(\sqrt{n})$ depth monotone circuits, but it is a computed by a $O(\log^2 n)$ depth circuit having only **two** gates with non-zero orientation.

Uniform orientation

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For Clique, consider β as a $\sqrt{n} \times \sqrt{n}$ matrix.

A "symmetric square" is a rectangle indexed by the same set of vertices.
Structure Based Lower Bound: Case of **Clique**

If C computes **Clique** :

$$\left\{ \begin{array}{l} \beta\text{-matrix has a 0-symm-sq.} \\ \text{of order } O(\log^{1+\epsilon} n) \end{array} \right\} \Longrightarrow \left\{ \text{Depth must be } \omega(\log n) \right\}$$

In contrast : Let U be symmetric square of order $O(\log n)$.
If C computes **Clique**:

$$\left\{ \text{Depth } d \right\} \Longrightarrow \left\{ \begin{array}{l} \text{Depth } d + c(\log n) \\ U \text{ is all 0s in the } \beta\text{-matrix.} \end{array} \right\}$$

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Proof Sketch : KW Games

Alice is given $x \in f^{-1}(1)$ and Bob is given $y \in f^{-1}(0)$.

KW(f) : **Goal** : Find $i \in [n]$ such that $x_i \neq y_i$.
KW$(f) = \mathit{Depth}(f)$.

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If f is montone :
$\mathbf{KW}^+(f)$: **Goal** : Find $i \in [n]$ such that $x_i = 1$ and $y_i = 0$

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Known Lower Bounds :

- **KW**$^+$(**Clique**) $= \Omega(\sqrt{n})$.
- **KW**$^+$(s-t connectivity) $= \Omega(\log^2 n)$.
- **KW**$^+$(Perf. Match) $= \Omega(\sqrt{n})$.

Tradeoff : Depth Lower Bound vs Weight

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Let C be a weight w-restricted circuit computing a monotone function $f : \{0,1\}^n \rightarrow \{0,1\}$, then

$$\text{Depth}(C) = \Omega\left(\frac{\mathbf{KW}^+(f)}{4w+1}\right)$$

Proof Sketch

From Circuits to KW games: Let C be a monotone circuit computing f.
Alice is given $x \in f^{-1}(1)$ and Bob is given $y \in f^{-1}(0)$ for a monotone function f. **Goal** : Find $i \in [n]$ such that $x_i = 1$ and $y_i = 0$.

- Protocol : Top-down. Current gate g with inputs g_1 and g_2.
- Invariant at a gate g : $g(x) = 1$ and $g(y) = 0$.
- if g is $\vee$ gate, Alice sends 0 if $g_1(x) = 1$ else 1.
- if g is $\wedge$ gate, Bob sends 0 if $g_1(x) = 0$ else 1.

Proof Sketch

At a gate g whose orientation is $\beta \in \{0,1\}^n$

Subcube-Monotonicity Invariant : Restricted to the sub-cube outside the current β, the function g is monotone. Be within such a subcube.

Procotol : By using $2w$ bits of communication :

- Either conclude that there is an index i (where $\beta_i = 1$) such that $x_i = 1$ and $y_i = 0$, OR
- Change x and y to new pair x' and y' such that on bits indexed by β they agree, and $g(x') = 1$ and $g(y') = 0$.

How do we do the second step? Construct y' by setting $y_\beta = x_\beta$.

Since we know that $x_\beta \leq y_\beta$, "decreasing" y to y' will not make the function value of g as 1.

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Handling negation gates : Observe that negation gates can depend on at most $2w$ inputs.

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Future Work and Open problems

- Can we push the boundary beyond $O(\frac{\sqrt{n}}{\log^{1+\epsilon} n})$?
- Can we reduce "weight of orientation" in general (when we know the function computed is a monotone function)?
- Is there a structure vs weight trade-off?
- Can this new measure help in learning restricted non-monotone circuits?

Thanks !!