Aligning Individual and Societal Interests in Broadcast Games Via Subsidies

John Augustine

(joint work with I. Caragiannis, A. Fanelli, and C. Kalaitzis)
Reducing Maintenance Cost of Software Technologies

Software Technologies

Projects

John Augustine

Subsidies in Broadcast Games
Reducing Maintenance Cost of Software Technologies

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Projects

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Subsidies in Broadcast Games
Outline of the Talk

- Game theoretic **preliminaries**
- Broadcast Games
- Can we find a state that is “good” AND “stable”? No!
- Can we “stabilize” a good solution via subsidies? Yes and No!
- Plan for the future
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Strategic Game

Set $N = \{1, 2, \ldots, n\}$ of players.
- Rational (has an objective and seeks it)
- Intelligent (and knowledgeable about game)
- Non-cooperative

$\Sigma_i$ is the set of strategies for player $i$.

The state $S$ of the game is an element in the state space $\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$.

Each player incurs a cost $c_i(S)$ when the game is in state $S$. 
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**Example: Rock-Paper-Scissors**

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>(0.5, 0.5)</td>
<td>(1, 0)</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>Paper</td>
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</table>

Row player pays 0
Col player pays 1
Nash Equilibrium

Unilateral Move:
- Consider a state $S = (s_1, s_2, \ldots, s_i, \ldots, s_n)$.
- We use $(S_{-i}, s_i')$ to denote the state $(s_1, s_2, \ldots, s_i', \ldots, s_n)$.

Improvement Move

An improvement move of player $i$ in state $S = (s_1, s_2, \ldots, s_n)$, is a strategy $s_i'$ such that $c_i(S_{-i}, s_i') < c_i(S)$. 
Nash Equilibrium

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(Pure) Nash Equilibrium (NE)

A game state such that no player has an improvement move.

- Simple but fundamental notion of a stable state.
- Some caveats:
  - It does not always exist. E.g., rock-paper-scissors.
  - Not necessarily good for the players.
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Congestion Games — formal definition

- \( N = \{1, 2, \ldots, n\} \), set of players.
- \( E = \{e_1, e_2, \ldots, e_m\} \), set of resources.
  - \( w(e) \) is the weight or cost of resource \( e \in E \).
- \( \Sigma_i \subseteq 2^E \), set of strategies of player \( i \).
  - Data center example.
  - All paths from \( s \) to \( t \).
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Subsidies in Broadcast Games
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Congestion Games — formal definition

- $f_e(k)$, payment incurred by each of the $k$ players using $e$.
  - In *cost sharing* congestion games, $f_e(k) = \frac{w(e)}{k}$.

- $c_i(S) = \sum_{e \in S_i} f_e(n_e(S))$, cost function of player $i$.
  - $n_e(S) =$ number of players using $e$ in $S$.

**Congestion games ≡ exact potential games.**

Exact potential games always admit Nash equilibria.
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**Congestion games $\equiv$ exact potential games.**

A Simple Network Design Example

All Edges Cost 1 and $f_e(k) = \frac{1}{k}$
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Classes of Games — Recap

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- Strategic Games
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- Strategic Games
- Congestion Games
- Network Design Games
Classes of Games — Recap

- **Strategic Games**
  - **Congestion Games**
    - **Network Design Games**
      - **Cost Sharing ND Games**
Classes of Games — Recap

Strategic Games

Congestion Games

Network Design Games

Cost Sharing ND Games

Broadcast Games
Broadcast Games
Broadcast Games

Edge $e$ of weight $w(e)$ to root vertex
Broadcast Games

Every player needs a path to the root.
Broadcast Games

The 6 players who use $e$ pay $\frac{w(e)}{6}$ for $e$.

Every player needs a path the root.
Broadcast Games

Recall: players are rational and intelligent.

Consider a **Nash equilibrium**

Can the edges used by players form a cycle? **NO!**
Recall: players are rational and intelligent.

**Consider a Nash equilibrium**

Can the edges used by players form a cycle? \textbf{NO!}
Are all Nash equilibria equally good? NO

Social Cost of state $S$ is $C(S) = \sum_i c_i(S)$

Let $OPT = \min_S C(S)$

What is $OPT$ for Broadcast games?

Price of Anarchy (PoA)

Let $\mathcal{N}$ the set of all NE.

$$\text{PoA} = \frac{\text{Social Cost of worst NE}}{OPT} = \max_{s \in \mathcal{N}} \frac{C(S)}{OPT}$$
Price of Anarchy

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A Simple Network Design Example

All Edges Cost 1 and \( f_e(k) = \frac{1}{k} \)

Price of Anarchy = \( \frac{6}{?} \)
A Simple Network Design Example

All Edges Cost 1 and $f_e(k) = \frac{1}{k}$

Price of Anarchy $= \frac{6}{5}$
Bad Example for Price of Anarchy

Price of Anarchy $= n$ even in Broadcast games!

$n$ players

All Edges in this part have $w(\cdot) = 0$

Each player wants a path to $t$

$w(\cdot) = n$

$w(\cdot) = 1$
Bad Example for Price of Anarchy

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Price of Anarchy = $n$ even in Broadcast games!
Benevolent central authority wants to “recommend” a stable state that is good for everybody.

- Is there always an equilibrium that is also socially optimal?
- Unfortunately NOT!
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Price of Stability
Price of Stability (PoS)

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Cost of Players Along a Path

Player $n$ incurs $H_n$

Player $n - k$ incurs $H_n - H_k$

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Subsidies in Broadcast Games
Our Setting

- Benevolent central authority.
  - wants to suggest a solution that is
    - optimal and
    - stable, i.e., a Nash equilibrium
  - She is even willing to spend money, but
    - only if necessary
    - minimize amount spent
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Our Results

**Question 1:** Is there an **optimal AND stable** solution?

Is PoS=1?

- **YES** ⇒ ∃ a stable optimal solution (MST).
- **NO** ⇒ ∅ any stable MST.
- NP-complete
- We also show that it is APX-hard to approximate PoS.

- We can easily find an MST, but **cannot find a stable MST.**
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Our Results — The Power of Subsidies

- We consider the option of subsidies.

- In classic cost sharing, each player pays \( \frac{w(e)}{k} \).

- When a subsidy \( b \) is placed on \( e \), each player pays \( \frac{w(e) - b}{k} \).

Players can be incentivized to follow a path by subsidizing its edges.
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Our Results

Question 2: Given an MST $T$, how to stabilize it via subsidies?

- When subsidies can be fractional
  - Linear programming formulation
  - The total amount needed to subsidize $T$ is at most $\frac{MST}{e}$
  - $\exists$ simple tight examples.

- When subsidies must be all-or-nothing
  - Cannot be approximated within any arbitrary constant
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Related Works that use Subsidies

Challenge: descriptive $\rightarrow$ prescriptive

- Monderer and Tennenholtz (2003): VCG auctions. (They call it $k$-implementation.)
- Eidenbenz, Oswald, Schmid, and Wattenhofer (2007): Mechanism Design. (They study a malicious central authority.)
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“Exact Bin Packing” $\rightarrow$ “Is PoS = 1”?
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Given $\sum_i s_i = kB$, can we “exactly” pack items in bins?
Given $\sum_i s_i = kB$, can we “exactly” pack items in bins?

**Strongly NP-Complete**
Choose large $\ell$ so that $H_{B+\ell} - H_B > 1$.

- Therefore $\text{BYPASS} \notin \text{MST}$.

- If $\beta < B$, then $b$ takes $\text{BYPASS}$ edge.
- Otherwise, if $\beta \geq B$, $b$ takes the MST path.
Choose large $\ell$ so that $H_{B+\ell} - H_B > 1$.

Therefore Bypass $\not\in$ MST.

If $\beta < B$, then $b$ takes Bypass edge.

Otherwise, if $\beta \geq B$, $b$ takes the MST path.
The Bypass gadget

Choose large $\ell$ so that $H_{B+\ell} - H_B > 1$.

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Is PoS = 1?
Computing PoS is APX-hard

Constructing a Broadcast Game

$H_{B+\ell} - H_B > 1$

Connector
Vertices
$s_1 - 1$
$s_2 - 1$
$s_3 - 1$
$\cdots$
$s_{n-1} - 1$
$s_n - 1$

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Subsidies in Broadcast Games
Constructing a Broadcast Game

Edges in this level have a uniformly high cost $c_{hi}$.

Edge costs 1

$H_{B+\ell} - H_B > 1$

Connector

Vertices

$x_1$ $x_2$ $x_3$ $x_{n-1}$ $x_n$
Reduction

- $C_{hi}$ is high so all the $s_i$ players corresponding to item $i$ can be forced to go up one edge.
  ⇒
- If $\exists$ exact packing,
- consider each $x_i$ connected to $b_j$ according to packing
- then every $b_j$ will get $B$ players from below
- No player will prefer the BYPASS edge.
  ⇐
- If a stable MST exists, then we can construct a packing.
Maximum Independent Set in 3-regular Graphs

- Given a 3-regular graph, what is the cardinality of the largest independent set of nodes?

- APX-hard (Berman and Karpinski, 1999).
Is PoS = 1? Computing PoS is APX-hard

MIS $\rightarrow$ Computing PoS
Is PoS = 1?
Computing PoS is APX-hard

MIS $\rightarrow$ Computing PoS

Unit cost edges to yellow and red nodes

Place new node on each edge.
MIS $\rightarrow$ Computing PoS

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Is PoS = 1?
Computing PoS is APX-hard

MIS $\rightarrow$ Computing PoS

Single Edge Branches
Is PoS = 1?
Computing PoS is APX-hard

MIS → Computing PoS

Triad Branches
Is PoS = 1?
Computing PoS is APX-hard

MIS $\rightarrow$ Computing PoS
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MIS → Computing PoS
MIS $\rightarrow$ Computing PoS

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Subsidies in Broadcast Games
Observation
A maximum independent set of size $k \iff$ cheapest Nash equilibrium has $k$ triad branches.

Theorem
*It is NP-hard to approximate the Price of Stability in broadcast games within a factor of $\frac{571}{570}$.***
Stabilizing a Minimum Spanning Tree

- Recall that an MST will be an optimal solution
- However, an MST may not be a Nash equilibrium

We ask:
Given a broadcast game on a graph $G$ and an MST $T$ of $G$, can we subsidize the edges of $T$ to ensure that it is stable?
Stabilizing a Minimum Spanning Tree
Stabilizing a Minimum Spanning Tree
Stabilizing a Minimum Spanning Tree

Red Player Also wants to deviate!
Stabilizing a Minimum Spanning Tree using LP

- Given a player $p$ who wants to deviate along $P$,
- the red player in $P$ is the player with exactly 1 cross edge and fewest tree edges.
- If $p$ wants to deviate along $P$, then the red player in $P$ also wants to deviate.
- Contrapositive: If red player does not want to deviate, then $p$ does not want to deviate along $P$. 

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Stabilizing a Minimum Spanning Tree using LP

- Let $x_e$ be the subsidy on $e$. $\vec{x}$ is the vector of subsidies.
- Objective: Minimize $\sum_{e \in T} x_e$.
- Assume all players follow the given MST $T$.
- Let $c_i(\vec{x})$ be the cost paid by player $i$ under subsidy vector $\vec{x}$.
- $\forall$ players $i$ and $j \in Ancest_T(i) \cap Neighbor_G(i)$
  
  Constraint: $c_i(\vec{x}) \leq w(e) + c_j(\vec{x})$

- Let $c^i_j(\vec{x})$ be the cost paid by player $j$ under subsidy vector $\vec{x}$ when player $i$ passes through $j$.
- $\forall$ players $i$ and $j \in Neighbor_G(i) \setminus (Ancest_T(i) \cup Desc_T(i))$
  
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Stabilizing a Minimum Spanning Tree using LP

Minimize $\sum_{e \in T} x_e$.

Subject to:

$\vec{x} \geq 0$

For every player $i$ and $j \in \text{Neighbor}_G(i)$

$c_i(\vec{x}) \leq w(e) + c_j(\vec{x})$ if $j \in \text{Ancest}_T(i)$

$c_i(\vec{x}) \leq w(e) + c_j^i(\vec{x})$ if $j \notin \text{Ancest}_T(i) \cup \text{Desc}_T(i)$

Skip bounds on fractional subsidies.
Can we bound the total amount of subsidies in the worst case?

If an internal player can deviate, then $n$ can deviate.

Contrapositive: If $n$ cannot deviate, then no other player can deviate!
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**Contrapositive:** If $n$ cannot deviate, then no other player can deviate!
The best solution is to subsidize bottom $k$ edges as much as is required.

When subsidized, we need to ensure that

$$c_n(\cdot) = H_n - H_k \leq 1$$
Lower Bound on Subsidies

\[ 1 \geq H_n - H_k \]

\[ \approx \ln n - \ln k = \ln \frac{n}{k} \]

\[ \therefore \quad k \geq \frac{n}{e} - 2 = \frac{MST}{e} - 2 \]

(Approximations are for presentation purpose only.)
Upper Bound on Subsidies — Single Path

Every edge $e$ has $w(e) = 1$
Upper Bound on Subsidies — Single Path

Hard to work with actual costs
Need closed form expression

Every edge \( e \) has \( w(e) = 1 \)

\( n = 10 \)

John Augustine
Subsidies in Broadcast Games
Upper Bound on Subsidies — Single Path

Every edge $e$ has $w(e) = 1$

$$VC(e) = \ln \frac{n_e}{n_e - 1 + b}$$

$$\ln \frac{3}{3 - 1 + 0} = 0.405$$

John Augustine

Subsidies in Broadcast Games
Upper Bound on Subsidies — Single Path

When $b = 0$, $VC(e) > \frac{1}{n_e}$
When $b = 1$, $VC(e) = 0$

$VC(e) = \ln \frac{n_e}{n_e - 1 + b}$

$\ln \frac{3}{3-1+0} = 0.405$

Every edge $e$ has $w(e) = 1$
Upper Bound on Subsidies — Single Path

Virtual Cost (with one edge subsidized) of player $n = \text{sum of virtual costs of every edge}$

$$\sum_{i=2}^{n} \ln \left( \frac{i}{i-1} \right) + \ln \left( \frac{1}{1 - 1 + 1} \right) = \ln \left( \frac{n}{n-1} \frac{n-1}{n-2} \cdots \frac{2}{1} \right) = \ln \left( \frac{n}{1} \right)$$
Suppose we place total subsidies worth 2.5. We will place them on the least crowded edges.

Every edge $e$ has $w(e) = 1$. 

$n = 10$
Upper Bound on Subsidies — Single Path

Given that a total subsidies worth $b = 2.5$ is placed on the least crowded edges

Real Cost of player $n = \text{sum of her real costs on each edge}$

\[
= \frac{1}{10} + \frac{1}{9} + \frac{1}{8} + \frac{1}{7} + \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{2} + 0 \cdot 1 = 1.262
\]

Virtual Cost of player $n = \text{sum of her virtual costs on each edge}$

\[
= \ln \left( \frac{n}{b} \right) = \ln \left( \frac{10}{2.5} \right) = 1.386
\]
Upper Bound on Subsidies — Single Path

Algorithm to stabilize path:
compute and place subsidy $\bar{b}$ required to make virtual cost of player $n$ equal 1 on the least crowded edges.

$$\ln \left( \frac{n}{\bar{b}} \right) = 1. \quad \therefore \bar{b} = \frac{n}{e}$$

Proof of Correctness: If virtual cost is 1, real cost is at most 1
Upper Bound on Subsidies — General

- If some edges have weight 0, but others have weight 1, bound holds immediately.
- Extend to case where MST can be a tree and graph has weights in \{0, c\}, \(c > 0\).
- Extend to graphs with arbitrary weights by decomposing it.
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**All-or-Nothing Subsidies**

- Suppose edges can either be subsidized fully or left unsubsidized.
- Example: government maintains some road. Rest are maintained by users.

**Inapproximability Result**

It is NP-hard to approximate the optimal amount of total subsidies required within any constant factor.
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In this work, we only considered optimal solutions and pure Nash equilibria.

- approximate optimality
- $\epsilon$-Nash equilibria

General cost sharing network design games.

Subsidies to improve price of anarchy.
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Other Interests

I am interested in algorithms: online algorithms, approximation algorithms, computational geometry, and game theory.

Game theory topics:
- Congestion games
- Cut games

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THANK YOU!