Classical Cryptography

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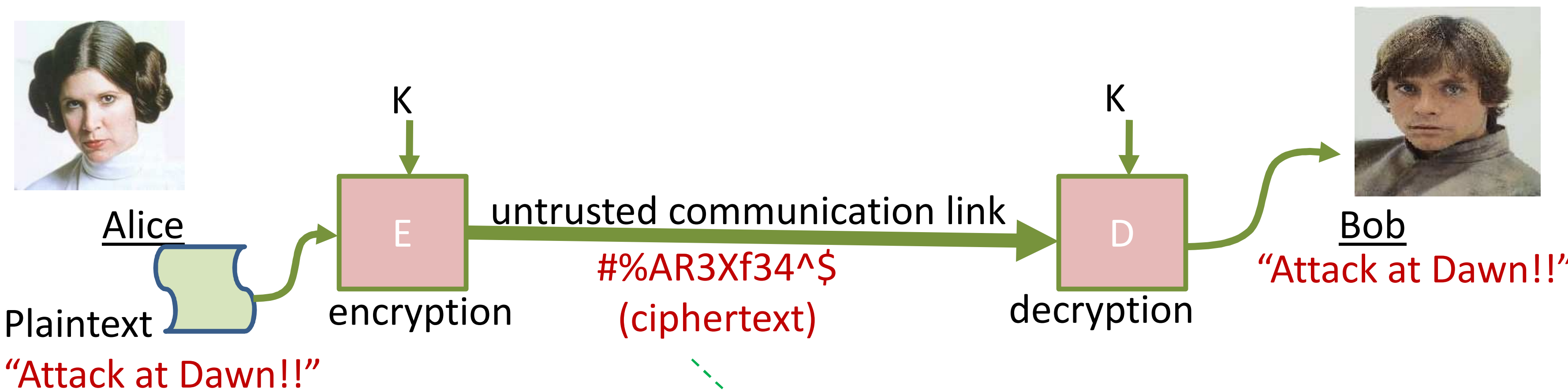
Ciphers

- Symmetric Algorithms
  - Encryption and Decryption use the same key
  - i.e. $K_E = K_D$
  - Examples:
    - Block Ciphers : DES, AES, PRESENT, etc.
    - Stream Ciphers : A5, Grain, etc.
- Asymmetric Algorithms
  - Encryption and Decryption keys are different
  - $K_E \neq K_D$
  - Examples:
    - RSA
    - ECC

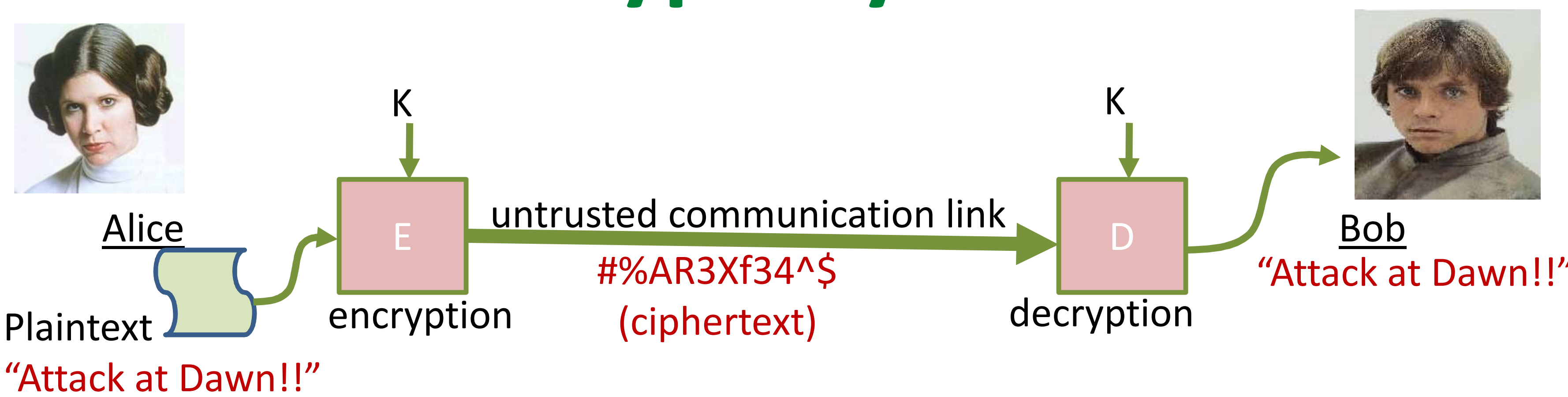
Encryption (symmetric cipher)



The Key K is a secret

Only sees ciphertext.
cannot get the plaintext message
because she does not know the key K

A CryptoSystem



A **cryptosystem** is a five-tuple $(\mathbb{P},\mathbb{C},\mathbb{K},\mathbb{E},\mathbb{D})$, where the following are satisfied:

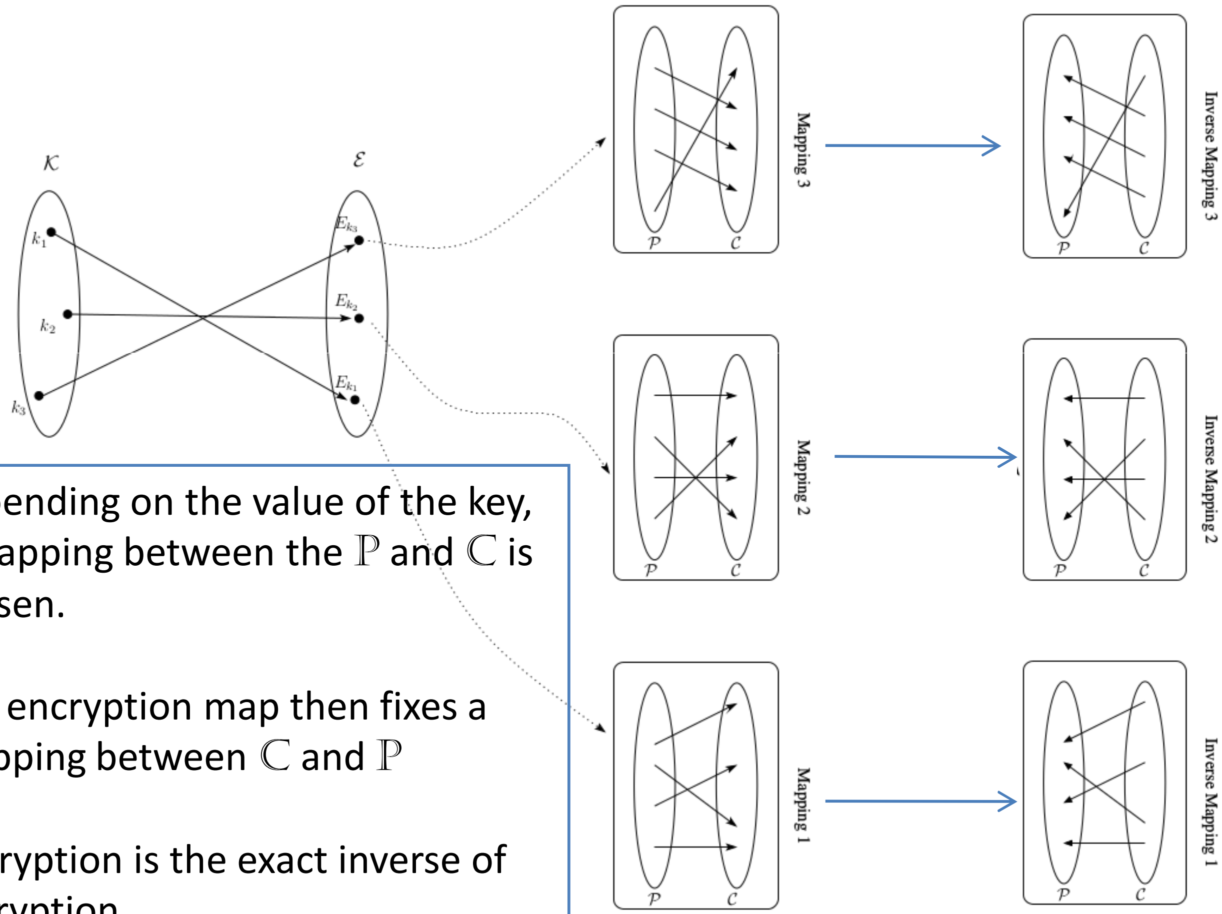
- $\mathbb{P}$ is a finite set of possible **plaintexts**
- $\mathbb{C}$ is a finite set of possible **ciphertexts**
- $\mathbb{K}$, the **keyspace**, is a finite set of possible **keys**
- $\mathbb{E}$ is a finite set of encryption functions
- D is a finite set of decryption functions
- $\forall K \in \mathbb{K}$

  Encryption Rule : $\exists e_K \in \mathbb{E}$, and

  Decryption Rule : $\exists d_K \in \mathbb{D}$

  such that $(e_K: \mathbb{P} \rightarrow \mathbb{C})$, $(d_k: \mathbb{C} \rightarrow \mathbb{P})$ and $\forall x \in \mathbb{P}$, $d_K(e_K(x)) = x$.

Pictorial View of Encryption



Depending on the value of the key, a mapping between the $\mathbb{P}$ and $\mathbb{C}$ is chosen.

The encryption map then fixes a Mapping between $\mathbb{C}$ and $\mathbb{P}$

Decryption is the exact inverse of encryption.

Attacker's Capabilities (Cryptanalysis)

Mallory wants to some how get information about the secret key.

- Attack models
  - ciphertext only attack
  - known plaintext attack
  - chosen plaintext attack

    Mallory has temporary access to the encryption machine. He can choose the plaintext and get the ciphertext.
  - chosen ciphertext attack

    Mallory has temporary access to the decryption machine. He can choose the ciphertext and get the plaintext.

Kerckhoff’s Principle for cipher design

- Kerckhoff’s Principle
  - The system is completely known to the attacker. This includes encryption & decryption algorithms, plaintext
  - only the key is secret

- Why do we make this assumption?
  - Algorithms can be leaked (secrets never remain secret)
  - or reverse engineered

history of A5/1: https://en.wikipedia.org/wiki/A5/1

Facts about e_K

- It is injective (one-to-one)
  - i.e. $e_k(x_1) = e_k(x_2)$ iff $x_1 = x_2$
  - Why?
    - If not, then Bob does not know if the ciphertext came from x_1 or x_2
- If $\mathbb{P} = \mathbb{C}$, then the encryption function is a permutation

  $\mathbb{C}$ is a rearrangement of $\mathbb{P}$

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A Shift Cipher

- Plaintext set : $\mathbb{P}$ = {0,1,2,3 ..., 25}
- Ciphertext set : $\mathbb{C}$ = {0,1,2,3 ..., 25}
- Keyspace : $\mathbb{K}$ = {0,1,2,3 ..., 25}
- Encryption Rule : $e_K(x) = (x + K) \bmod 26$,
- Decryption Rule : $d_k(x) = (x - K) \bmod 26$

  where $K \in \mathbb{K}$ and $x \in \mathbb{P}$
- *Note:*
  – *Each K results in a unique mapping* $e_K: \mathbb{P} \to \mathbb{C}$ and $d_K: \mathbb{C} \to \mathbb{P}$
  – $d_k(e_K(x)) = x$
  – *The encryption/decryption rules are permutations*

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Using the Shift Cipher

with K=3

	0	1	2	3	4	5	6	7	8	9	10	11	12
plaintext	a	b	c	d	e	f	g	h	i	j	k	l	m
ciphertext	D	E	F	G	H	I	J	K	L	M	N	O	P

	13	14	15	16	17	18	19	20	21	22	23	24	25
plaintext	n	o	p	q	r	s	t	u	v	w	x	y	z
ciphertext	Q	R	S	T	U	V	W	X	Y	Z	A	B	C

attackatdawn $\longrightarrow$ DWWDFNDWFDZQ

Shift Cipher Mappings

- Each K results in a unique mapping $e_K: \mathbb{P} \to \mathbb{C}$ and $d_K: \mathbb{C} \to \mathbb{P}$
- The mappings are injective (one-to-one)

$y_1, y_2 \in \mathbb{C}$
$d_K(y_1) \neq d_K(y_2)$

plaintext	**a**	**b**	**c**	**d**	**…**	**x**	**y**	**z**
	0	1	2	3		23	24	25
	K=8							
ciphertext	8	9	10	11		5	6	7
	I	**J**	**K**	**L**		**F**	**G**	**H**
	K=10							
ciphertext	10	11	12	13		7	8	9
	K	**L**	**M**	**N**		**H**	**I**	**J**
	K=13							
ciphertext	13	14	15	16		10	11	12
	N	**O**	**P**	**Q**		**K**	**L**	**M**

Encryption Rule

$e_K(x) = (x + K) \bmod 26,$

Decryption Rule

$d_k(x) = (x - K) \bmod 26$

How good is the shift cipher?

- A good cipher has two properties
  - Easy to compute
    - Satisfied
  - An attacker (Mallory), who views the ciphertext should not get any information about the plaintext.
    - Not Satisfied!!
    - The attacker needs at-most 26 guesses to determine the secret key ….
      - This is an exhaustive key search (known as brute force attack)

Puzzle

- Cryptanalyze, assuming a shift cipher

"COMEBSDISCKCCDBYXQKCSDCGOKUOCDVSXU"

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Cryptanalysis of Shift Cipher

By Brute Force...

Ciphertext : "DWWDFNDWGDZQ"

- There are only 26 possible keys, so 26 possible decryptions
- Try all of them
  - key=0, "dwwdfndwgdzq"
  - key=1, "cvvcemcvfcyp"
  - key=2, "buubdlbuebxo"
  - key=3, "attackatdawn" . . . makes sense
  - key=4, . . .
  - key=25, . . .
- Only key=3 makes sense, thus it is likely to be the key
- ... too easy!!!

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History & Usage

- Used by Julius Caesar in 55 AD with K=3. This variant known as Caesar's cipher.
- Augustus Caesar used a variant with K=-1 and no mod operation.
- Shift ciphers are extremely simple, still used in Modern times
  - By Russian Soldiers in first world war
  - Last known use in 2011 (by militant groups)

Interesting Read: https://en.wikipedia.org/wiki/Caesar_cipher

Substitution Cipher

- Plaintext set : $\mathbb{P}$ = {a,b,c,d,…,z}
- Ciphertext set : $\mathbb{C}$ = {A,B,C,D,…,Z}
- Keyspace : $\mathbb{K}$ = {π | such that π is a permutation of the alphabets}
  - Size of keyspace is 26!
- Encryption Rule : $e_\pi(x) = \pi(x)$,
- Decryption Rule : $d_\pi(x) = \pi^{-1}(x)$

Substitution Cipher Example

Key is some permutation of the alphabets

plaintext	a	b	c	d	e	f	g	h	i	j	k	l	m
ciphertext	Z	J	H	K	F	G	I	D	B	E	A	C	P

plaintext	n	o	p	q	r	s	t	u	v	w	x	y	z
ciphertext	Y	M	S	L	V	N	X	O	T	R	Q	U	W

Plaintext : "attackatdawn"

Ciphertext : "ZXXZHAXKZRY"

26! permutations possible. Thus possible keys are $26! \approx 4 \times 10^{26}$ .... rules out brute force!!!

Note that the shift cipher is a special case of the substitution cipher which includes only 26 of the 26! keys

Cryptanalysis of Substitution Cipher (frequency analysis)

Languages do not have uniform probabilities

- Unigram probabilities of alphabets
  - E has probability 0.12 (12%)
  - T,A,O,I,N,S,H,R each have probabilities between 0.06 and 0.09
  - D,L each have probabilites around 0.04
  - C,U,M,W,F,G,Y,P,B each have probabilities between 0.015 and 0.028
  - V,K,J,X,Q,Z each occur less than 0.01
- 30 common digrams are TH, HE, IN, ER, AN, RE, AT,. . .

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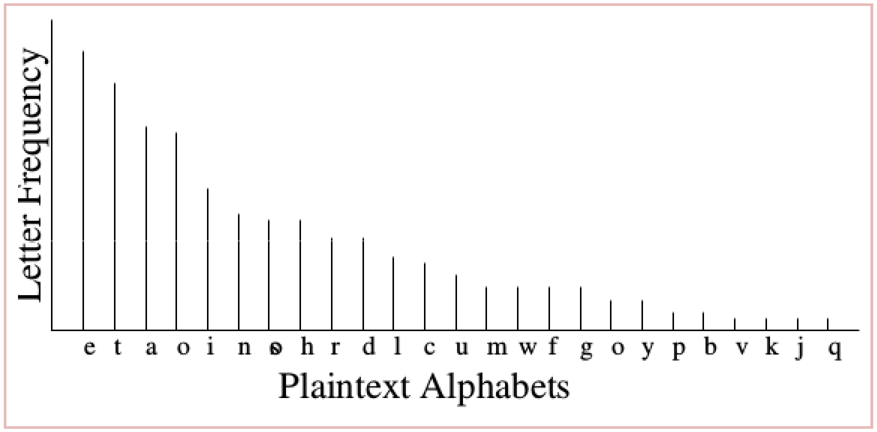
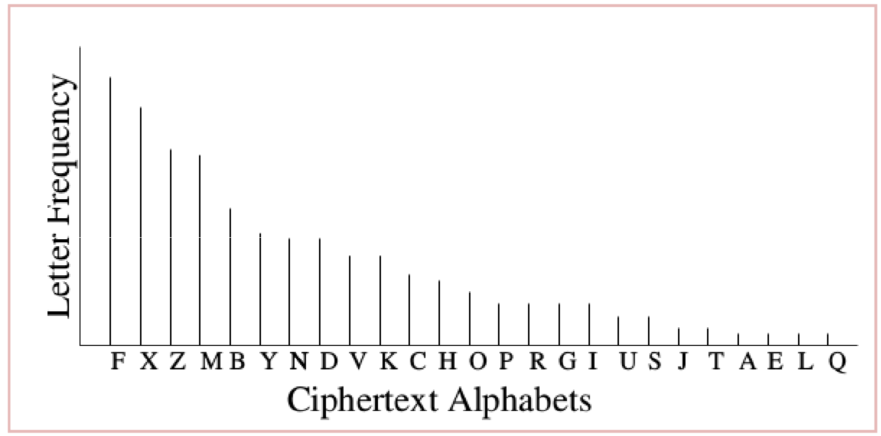
Cryptanalysis of Substitution Cipher *(from their frequency characteristics)*



Frequency analysis of plaintext alphabets



Frequency analysis of ciphertext alphabets

Usage & Variants

- Evidence showed that it was used before Caesar’s cipher
- The technique of ‘substitution’ still used in modern day block ciphers
- Frequency based analysis attributed to Al-kindi, an Arab mathematician (in AD 800)

Polyalphabetic Ciphers

- Problem with the simple substitution cipher :
  - A plaintext letter always mapped to the same ciphertext letter

    eg. ‘Z’ always corresponds to plaintext ‘a’
  - facilitating frequency analysis

- A variation (polyalphabetic cipher)
  - A plaintext letter may be mapped to multiple ciphertext letters
  - eg. ‘a’ may correspond to ciphertext ‘Z’ or ‘T’ or ‘C’ or ‘M’
  - More difficult to do frequency analysis (but not impossible)

  - Example : Vigenere Cipher, Hill Cipher

Vigenère Cipher

- Let the key be (2,5,8,7,9,12) of size 6
- Let the message to be encrypted be "attackatdawn"
- Convert message to integers modulo 26
  - "attackatdawn" becomes (0, 19, 19, 0, 2, 10, 0, 19, 3, 0, 22, 13)
- To encrypt, group them in terms of 6 and add the corresponding key

|keyspace| = 26^m
(where m is the length of the key)

plaintext (x)	a	t	t	a	c	k	a	t	d	a	w	n
	0	19	19	0	2	10	0	19	3	0	22	13
key (k)	2	5	8	7	9	12	2	5	8	7	9	12
(x + k) mod 26	2	23	1	7	11	22	2	24	11	7	[illegible]	25
ciphertext	C	X	B	I	K	W	C	Y	K	H	F	Z

Cryptanalysis of Vigenère Cipher

- Frequency analysis more difficult
  (but not impossible)
- Attack has two steps

1. Determine the length m of the key
2. Determine $K = (k_1, k_2, k_3, \cdots k_m)$ by finding each k_i separately

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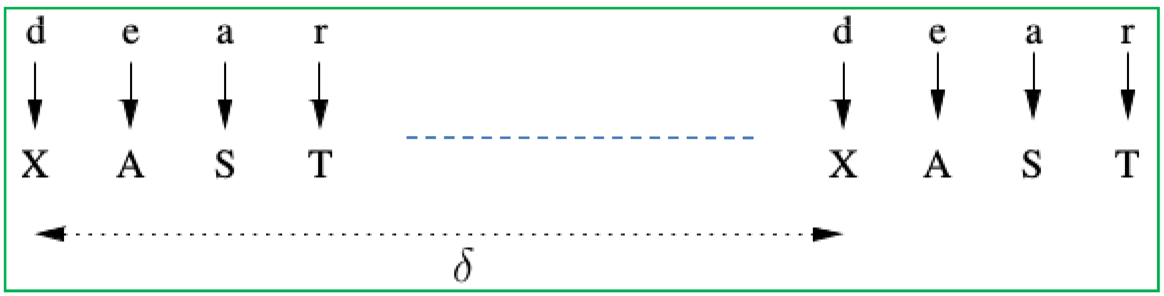
Determining Key Length *(Kaisiki Test)*

- **Kasiski test** by Friedrich Kasiski in 1863
- Let *m* be the size of the key
- observation: two identical plaintext segments will encrypt to the same ciphertext when they are δ apart and $(m \mid \delta)$



when (m divides δ)

- If several such δs are found (i.e. $\delta_1, \delta_2, \delta_3, \ldots.$) then
  - $m|\delta_1, m|\delta_2, m|\delta_3, \ldots.$
  - Thus m divides the gcd of $(\delta_1, \delta_2, \delta_3, \ldots.)$

Increasing Confidence of Key Length *(Index of Coincidence)*

- Consider a multi set of letters of size N

  say ***s*** *= {a,b,c,d,a,a,e,f,e,g,.....}*

- Probability of picking two 'a' characters (without replacement) is

$$\boxed{\frac{n_0}{N} \times \frac{n_0 - 1}{N - 1}}$$

probability the first pick is 'a'

probability the second pick is 'a'

n_0 : Number of occurrences of 'a' in S

- Sum of probabilities of picking two similar characters is

$$\boxed{I_c = \sum_{i=0}^{25} \frac{n_i(n_i - 1)}{N(N-1)}}$$

index of coincidence

Index of Coincidence

- Consider a random permutation of the alphabets (as in the substitution cipher)

s = {a,b,c,d,a,a,e,f,e,g,.....} → *S = {X,M,D,F,X,X,Z,G,Z,J,.....}*

- Note that : $n_a = n_X$; thus the value of I_c remains unaltered

- Number of occurrence of an alphabet in a text depends on the language, thus each language will have a unique I_c value

English	0.0667	French	0.0778
German	0.0762	Spanish	0.0770
Italian	0.0738	Russian	0.0529

Index of Coincidence, NSA Declassified Document
https://www.nsa.gov/public_info/_files/friedmanDocuments/Publications/FOLDER_231/41760429079956.pdf

Modular Arithmetic

Modular Arithmetic

slides in Mathematical Background

Affine Cipher

- A special case of substitution cipher
- Encryption: $y = ax + b \pmod{26}$
- Decryption: $x = (y - b)a^{-1} \pmod{26}$
  - plaintext : $x \in \{0,1,2,3, \ldots. 25\}$
  - ciphertext : $y \in \{0,1,2,3, \ldots. 25\}$
  - key : (a,b)
    - where a and b $\in \{0,1,2,3, \ldots. 25\}$ and
    - $\gcd(a, 26) = 1$

why need this condition?

- Example: a=3, b=5
  - Encryption: x=4; y = (3*4 + 5)mod 26 = 17
  - Decryption: $x = (y - b)a^{-1} \bmod 26$

$a.a^{-1} = 1 \bmod 26$. The inverse exists only if a and 26 are prime

$a^{-1} = 9$ (Note that 3 * 9 mod 26 = 1)
(17 - 5)*9 mod 26 = 4

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why gcd(a,26) must be 1?

- Let *gcd(a, 26) = d > 1*
  - then *d|a* and *d|26*   *(i.e. d mod 26 = 0)*
  - y = ax + b mod 26

    Let ciphertext y = b ;   ax = 0 mod 26

    In this case x can have two decrypted values : 0 and d.

    Thus the function is not injective.... cannot be used for an encryption

What is the ciphertext when (1) x_1 = 1 and (2) x_2 = 14 are encrypted with the Affine cipher with key (4, 0)?

Usage & Variants of Affine Cipher

- Ciphers built using the Affine Cipher
  - Caesar's cipher is a special case of the Affine cipher with a = 1
  - Atbash
    - $b = 25$, $a^{-1} = a = 25$
    - Encryption : $y = 25x + 25 \bmod 26$
    - Decryption : $x = 25x + 25 \bmod 26$

Encryption function same as decryption function

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Hill Cipher

- Encryption: y = xK (mod 26)
- Decryption: x = yK^{-1} (mod 26)
  - plaintext : x $\in$ {0,1,2,3, …. 25}
  - ciphertext : y $\in$ {0,1,2,3, …. 25}
  - key : K is an invertible matrix
- example

$$K = \begin{bmatrix} 11 & 8 \\ 3 & 7 \end{bmatrix}$$

hill

$$K^{-1} = \begin{bmatrix} 7 & 18 \\ 23 & 11 \end{bmatrix}$$

$$K \bullet K^{-1} = 1 \bmod 26$$

plaintext

hill

(7,8)(11,11)

$$\begin{bmatrix} 7 & 8 \end{bmatrix} \times \begin{bmatrix} 11 & 8 \\ 3 & 7 \end{bmatrix} (\bmod 26) = \begin{bmatrix} 23 & 8 \end{bmatrix}$$

encryption

$$\begin{bmatrix} 23 & 8 \end{bmatrix} \times \begin{bmatrix} 7 & 18 \\ 23 & 11 \end{bmatrix} (\bmod 26) = \begin{bmatrix} 7 & 8 \end{bmatrix}$$

decryption

hill $\rightarrow$ (7,8)(11,11) $\longrightarrow$ (23,8)(24,9) $\rightarrow$ XIYJ

plaintext ciphertext

Cryptanalysis of Hill Cipher

- ciphertext only attack is difficult
- known plaintext attack

$$\underset{\text{known plaintext}}{(7,8)(11,11)} \times \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \longrightarrow \underset{\text{corresponding ciphertext}}{(23,8)(24,9)}$$

Form equations and solve to get the key

$$7k_{11} + 8k_{21} = 23 \qquad 7k_{12} + 8k_{22} = 8$$
$$11k_{11} + 11k_{21} = 24 \qquad 11k_{12} + 11k_{22} = 9$$

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Permutation Cipher

- Ciphers we seen so far were substitution ciphers
  - Plaintext characters substituted with ciphertext characters

kill $\longrightarrow$ XIYJ

plaintext ciphertext

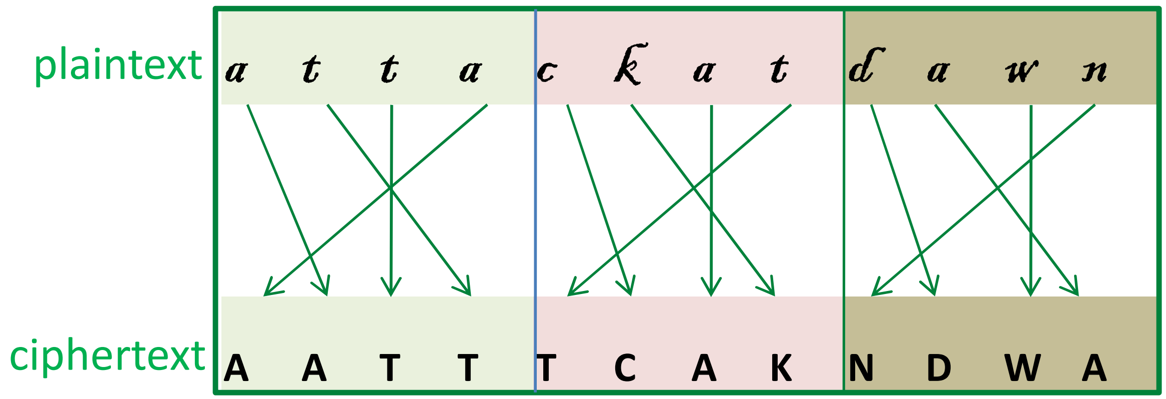
- Alternate technique : permutation
  - Plaintext characters re-ordred by a random permutation

kill $\longrightarrow$ LIHI

plaintext ciphertext

Permutation Cipher

- Example plaintext : *attackatdawn*
  - key : (1,3,2,0)   here is of length 4 and a permutation of (0,1,2,3)
    - It mean's 0^{th} character in plaintext goes to 1^{st} character in ciphertext (and so on…)



- cryptanalysis : 4! possibilities

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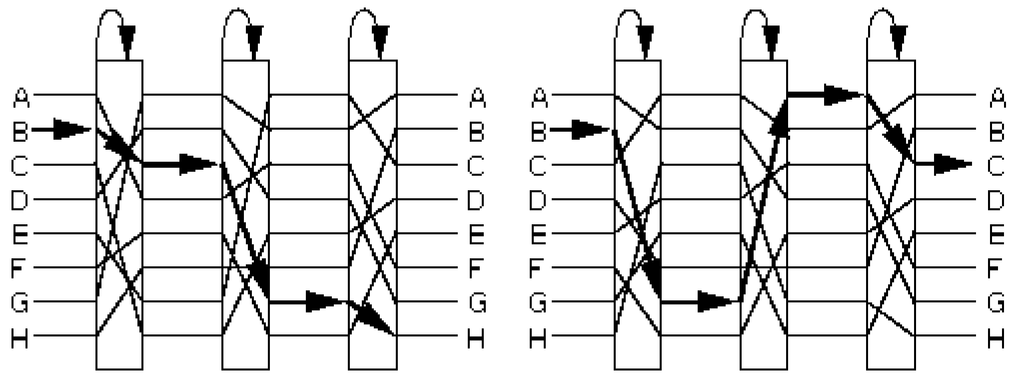
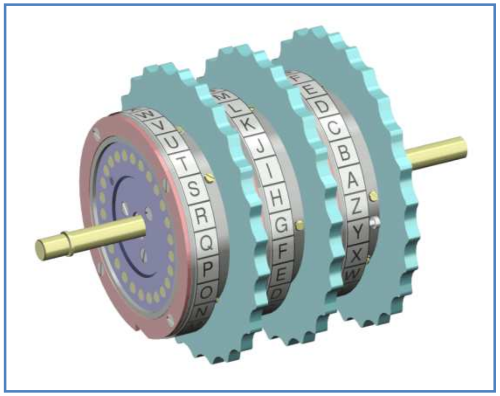
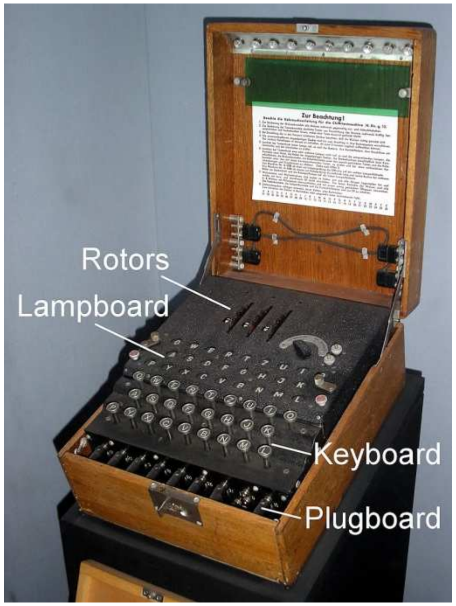
Rotor Machines (German Enigma)





- Each rotor makes a permutation
  - Adding / removing a rotor would change the ciphertext
- Additionally, the rotors rotates with a gear after a character is entered
- Broken by Alan Turing



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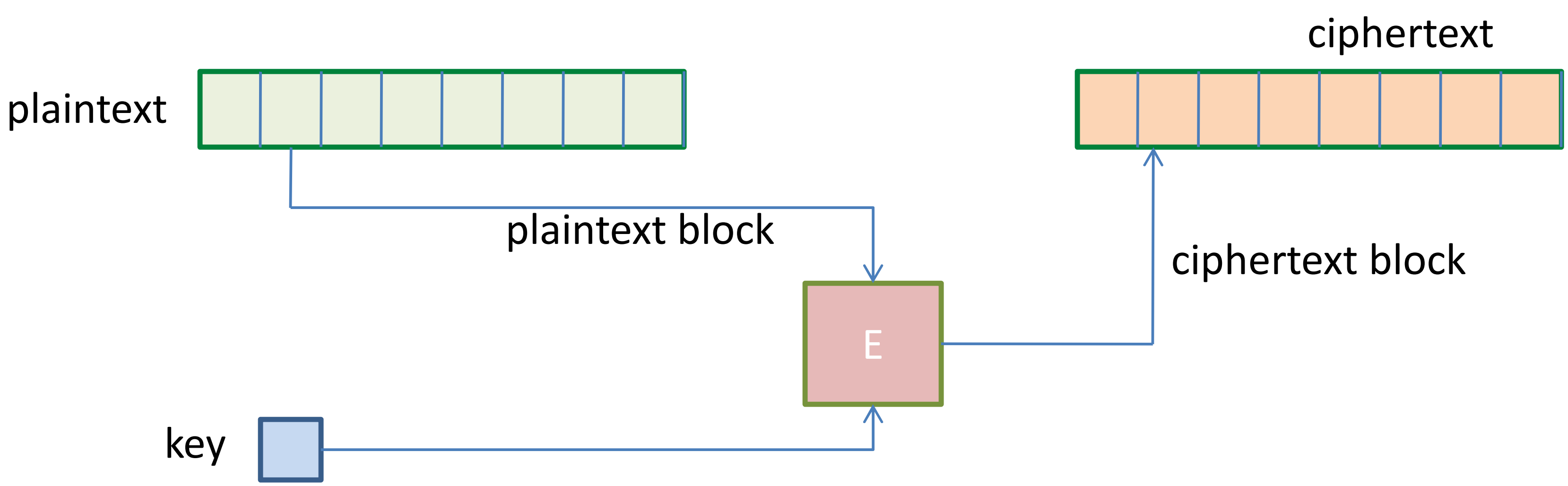
Block Ciphers

- General principal of all ciphers seen so far
  - Plaintext divided into blocks and each block encrypted with the same key
  - Blocks can vary in length starting from 1 character

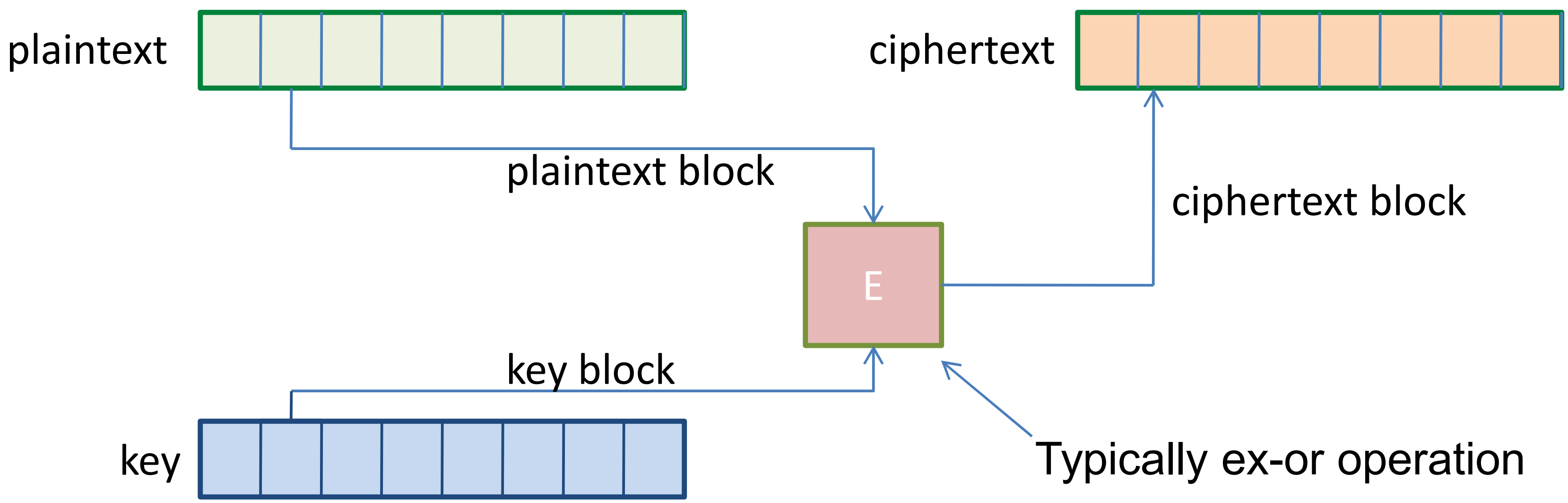


- examples: substitution ciphers, polyalphabetic ciphers, permutation ciphers, etc.

Stream Ciphers

Typically a bit, but can also more than a bit

- Each block of plaintext is encrypted with a different key



Formally, $y = y_1 y_2 y_3 \ldots = e_{k_1}(x_1) e_{k_2}(x_2) e_{k_3}(x_3) \ldots$

Observe: the key should be variable length… we call this a key stream.

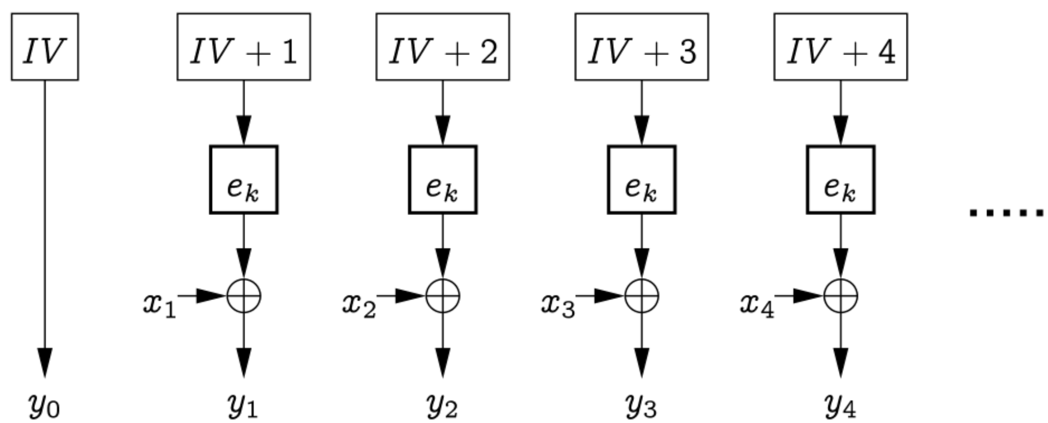
Stream Ciphers (how they work)

stream cipher output :
$$y = y_1 y_2 y_3 \cdots$$
$$y_1 = x_1 \oplus k_1 ;\ y_2 = x_2 \oplus k_2 ;\ y_3 = x_3 \oplus k_3 , \ldots.$$

How to generate the ith key : $k_i = f_i(K, k_1, k_2, k_3, \ldots, k_{i-1})$

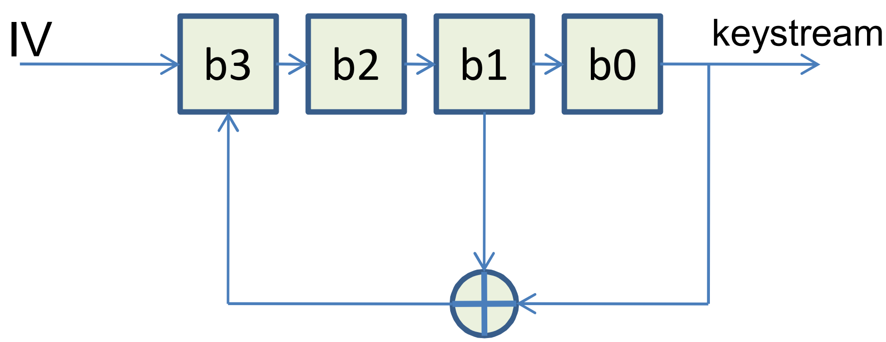
i^{th} key is a function of K and the first i-1 plaintexts

$k_1, k_2, k_3, \ldots, k_i$ Is known as the keystream



Generating the keystream in practice

- Using LFSRs (Linear feedback shift registers)



Initialization Vector

b3	b2	b1	b0
1	0	0	0
0	1	0	0
0	0	1	0
1	0	0	1
1	1	0	0
0	1	1	0
1	0	1	1
0	1	0	1
1	0	1	0
1	1	0	1
1	1	1	0
1	1	1	1
0	1	1	1
0	0	1	1
0	0	0	1
1	0	0	0

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Surprise Quiz-1

1. Prove that if the sum of all digits in a number is divisible by 9 then the number itself is divisible by 9.
2. How can the permutation cipher be represented as a Hill cipher? Explain with an example.
3. If GCD(a, N) = 1 then prove that a x i ≠ a x j mod N
4. Use (3) to show that a x k mod N is a permutation of {1,2, … N-1} where k varies from 1, 2, 3, …., N – 1.
5. Use (4) to show that the inverse of ‘a mod N’ (i.e. a^{-1}) exists (where gcd(a, N) = 1)

Credit will be given for whoever first puts up clear solutions in Google groups

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