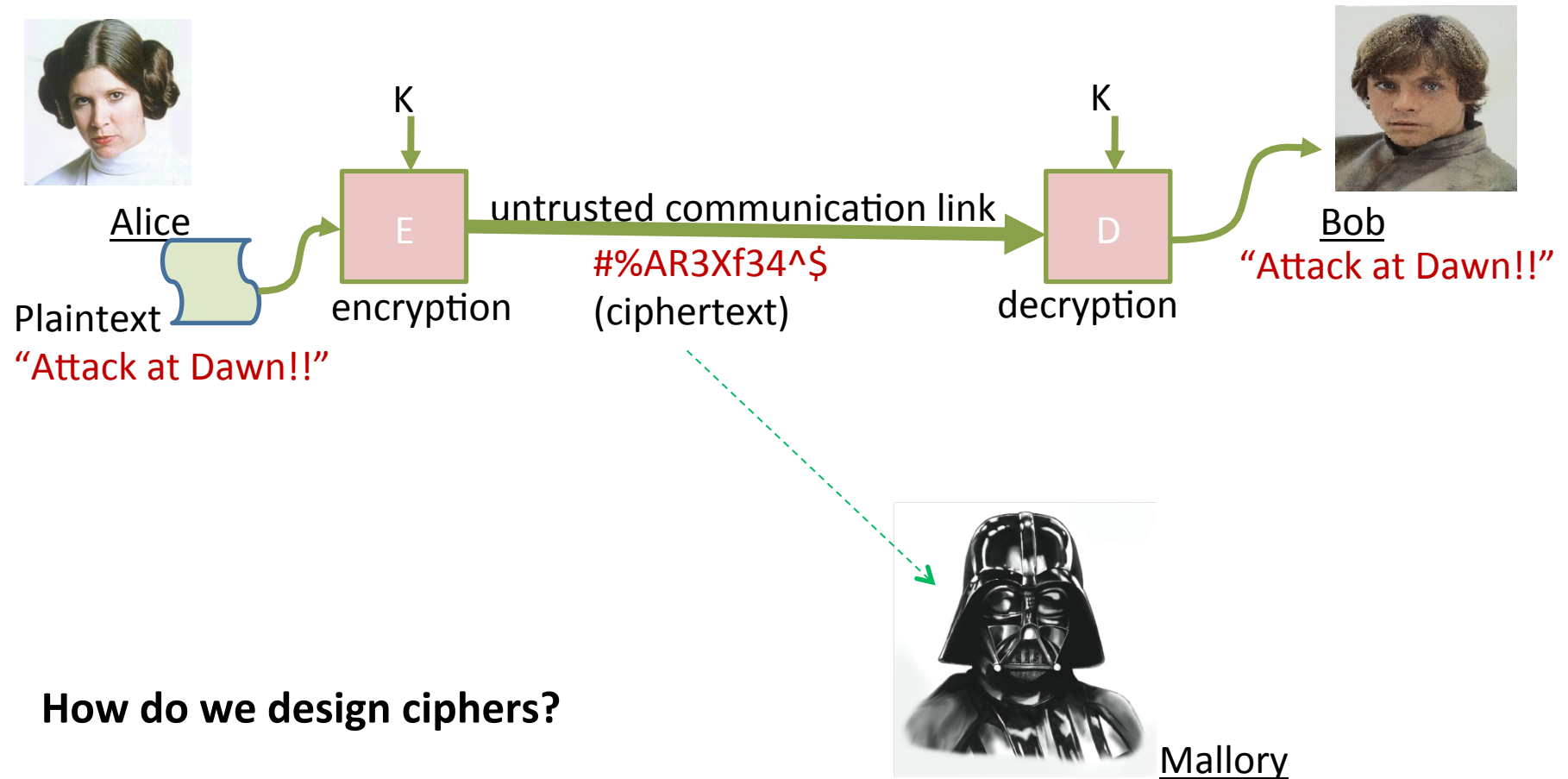


Perfect Secrecy

Chester Rebeiro

IIT Madras

Encryption



Cipher Models

(What are the goals of the design?)

Computation Security



My cipher can withstand all attacks with complexity less than 2^{2048}

The best attacker with the best computation resources would take 3 centuries to attack my cipher

Provable Security (Hardness relative to a tough problem)

If my cipher can be broken then large numbers can be factored easily



Unconditional Security



My cipher is secure against all attacks irrespective of the attacker's power. I can prove this!!

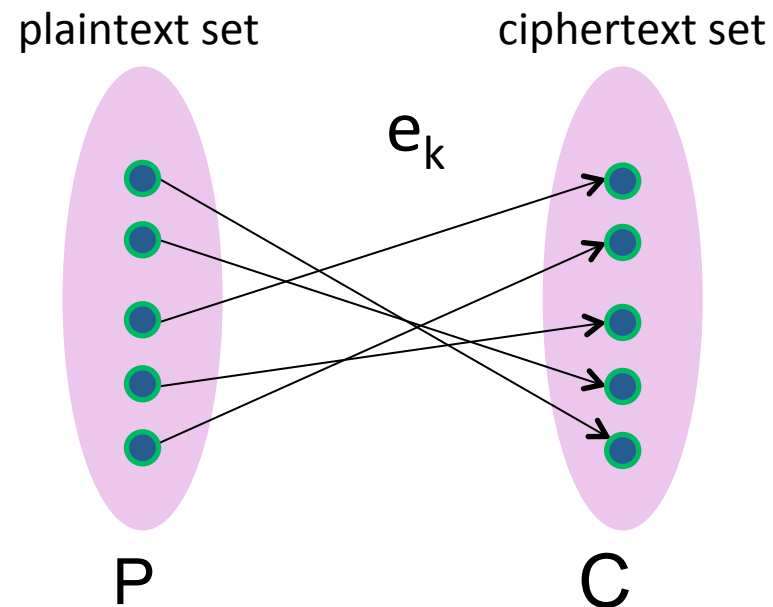
This model is also known as **Perfect Secrecy**. Can such a cryptosystem be built? We shall investigate this.

Analyzing Unconditional Security

- Assumptions
 - Ciphertext only attack model

The attacker only has information about the ciphertext. The key and plaintext are secret.
- We first analyze a single encryption then relax this assumption by analyzing multiple encryptions with the same key

Encryption

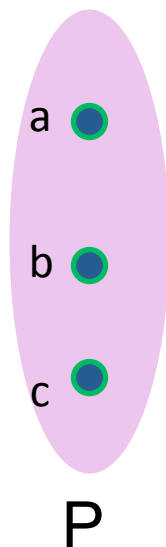


- For a given key, the encryption (e_k) defines an injective mapping between the plaintext set (P) and ciphertext set (C)
- Alice picks a plaintext $x \in P$, chooses a key (independently), and encrypts it to obtain a ciphertext $y \in C$

Plaintext Distribution

Plaintext Distribution

- Let \mathbf{X} be a discrete random variable over the set \mathbf{P}
- Alice chooses x from \mathbf{P} based on some probability distribution
 - Let $\Pr[\mathbf{X} = x]$ be the probability that x is chosen
 - This probability may depend on the language



Plaintext set

$$\Pr[\mathbf{X}=a] = 1/2$$

$$\Pr[\mathbf{X}=b] = 1/3$$

$$\Pr[\mathbf{X}=c] = 1/6$$

Note : $\Pr[a] + \Pr[b] + \Pr[c] = 1$

Key Distribution

Key Distribution

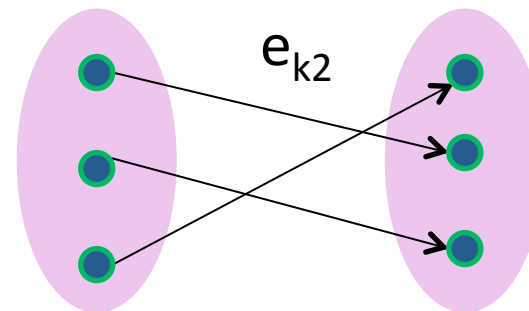
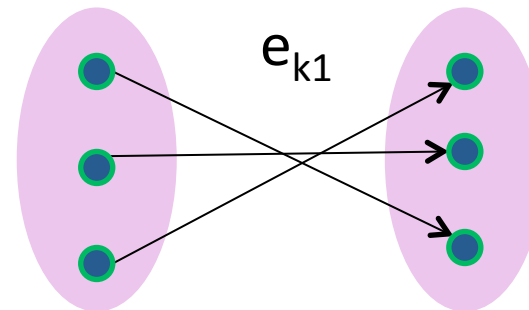
- Alice & Bob agree upon a key k chosen from a key set K
- Let K be a random variable denoting this choice

keyspace

$$\Pr[K=k_1] = \frac{3}{4}$$

$$\Pr[K=k_2] = \frac{1}{4}$$

There are two keys in the keyset
thus there are two possible encryption
mappings



Ciphertext Distribution

- Let \mathbf{Y} be a discrete random variable over the set \mathbf{C}
- The probability of obtaining a particular ciphertext y depends on the plaintext and key probabilities

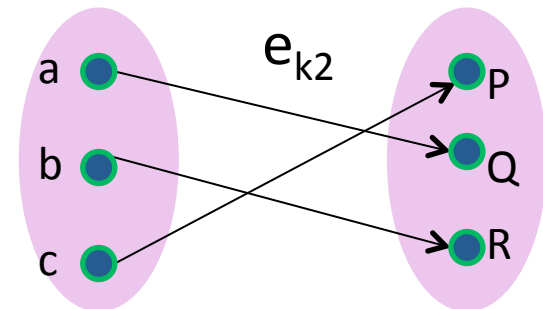
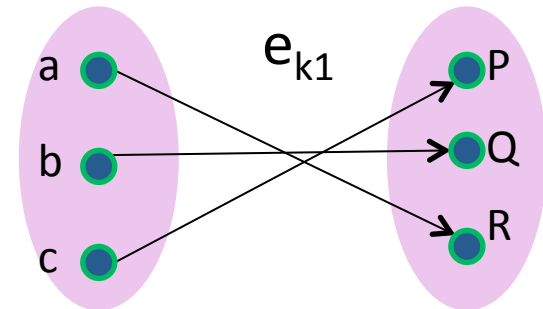
$$\Pr[Y = y] = \sum_k \Pr(k) \Pr(d_k(y))$$

$$\begin{aligned} \Pr[Y = P] &= \Pr(k_1) * \Pr(c) + \Pr(k_2) * \Pr(c) \\ &= (3/4 * 1/6) + (1/4 * 1/6) = \mathbf{1/6} \end{aligned}$$

$$\begin{aligned} \Pr[Y = Q] &= \Pr(k_1) * \Pr(b) + \Pr(k_2) * \Pr(a) \\ &= (3/4 * 1/3) + (1/4 * 1/2) = \mathbf{3/8} \end{aligned}$$

$$\begin{aligned} \Pr[Y = R] &= \Pr(k_1) * \Pr(a) + \Pr(k_2) * \Pr(b) \\ &= (3/4 * 1/2) + (1/4 * 1/3) = \mathbf{11/24} \end{aligned}$$

Note: $\Pr[Y=P] + \Pr[Y=Q] + \Pr[Y=R] = 1$



plaintext

$$\Pr[X=a] = 1/2$$

$$\Pr[X=b] = 1/3$$

$$\Pr[X=c] = 1/6$$

keyspace

$$\Pr[K=k_1] = 3/4$$

$$\Pr[K=k_2] = 1/4$$

Attacker's Probabilities

- The attacker wants to determine the plaintext x
- Two scenarios
 - Attacker does not have y (a priori Probability)
 - Probability of determining x is simply $Pr[x]$
 - Depends on plaintext distribution (eg. Language characteristics)
 - Attacker has y (a posteriori probability)
 - Probability of determining x is simply $Pr[x/y]$

A posteriori Probabilities

- How to compute the attacker's a posteriori probabilities? $\Pr[X = x \mid Y = y]$
 - Bayes' Theorem

$$\Pr[x \mid y] = \frac{\Pr[x] \times \Pr[y \mid x]}{\Pr[y]}$$

probability of the plaintext

probability of this ciphertext

?

The probability that y is obtained given x depends on the keys which provide such a mapping

$$\Pr[y \mid x] = \sum_{\{k : d_k(y)=x\}} \Pr[k]$$

$\Pr[y | x]$

$$\Pr[P | a] = 0$$

$$\Pr[P | b] = 0$$

$$\Pr[P | c] = 1$$

$$\Pr[Q | a] = \Pr[k_2] = \frac{1}{4}$$

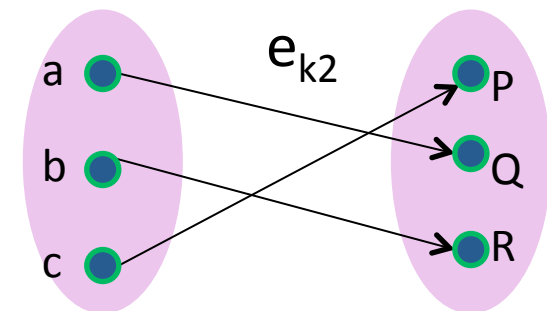
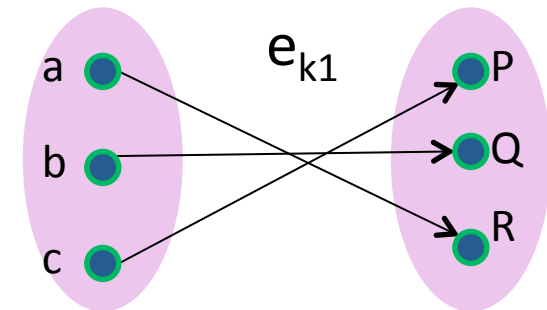
$$\Pr[Q | b] = \Pr[k_1] = \frac{3}{4}$$

$$\Pr[Q | c] = 0$$

$$\Pr[R | a] = \Pr[k_1] = \frac{3}{4}$$

$$\Pr[R | b] = \Pr[k_2] = \frac{1}{4}$$

$$\Pr[R | c] = 0$$



keyspace

$$\Pr[K=k_1] = \frac{3}{4}$$

$$\Pr[K=k_2] = \frac{1}{4}$$

Computing A Posteriori Probabilities

$$\Pr[x | y] = \frac{\Pr[x] \times \Pr[y | x]}{\Pr[y]}$$

plaintext	ciphertext	$\Pr[y x]$
$\Pr[X=a] = 1/2$	$\Pr[Y=P] = 1/6$	$\Pr[P a] = 0$
$\Pr[X=b] = 1/3$	$\Pr[Y=Q] = 3/8$	$\Pr[P b] = 0$
$\Pr[X=c] = 1/6$	$\Pr[Y=R] = 11/24$	$\Pr[P c] = 1$
		$\Pr[Q a] = 1/4$
		$\Pr[Q b] = 3/4$
		$\Pr[Q c] = 0$
		$\Pr[R a] = 3/4$
		$\Pr[R b] = 1/4$
		$\Pr[R c] = 0$

$$\Pr[a | P] = 0 \quad \Pr[b | P] = 0 \quad \Pr[c | P] = 1$$

$$\Pr[a | Q] = 1/3 \quad \Pr[b | Q] = 2/3 \quad \Pr[c | Q] = 0$$

$$\Pr[a | R] = 9/11 \quad \Pr[b | R] = 2/11 \quad \Pr[c | R] = 0$$

If the attacker sees ciphertext **P** then she would know the plaintext was **c**

If the attacker sees ciphertext **R** then she would know **a** is the most likely plaintext

Not a good encryption mechanism!!

Perfect Secrecy

- Perfect secrecy achieved when

a posteriori probabilities = a priori probabilities

$$\Pr[x | y] = \Pr[x]$$

i.e the attacker learns nothing from the ciphertext

Intuitively, by seeing the safe, you learn nothing about what is in it



Perfect Secrecy Example

- Find the a posteriori probabilities for the following scheme
- Verify that it is perfectly secret.

plaintext

$$\Pr[X=a] = 1/2$$

$$\Pr[X=b] = 1/3$$

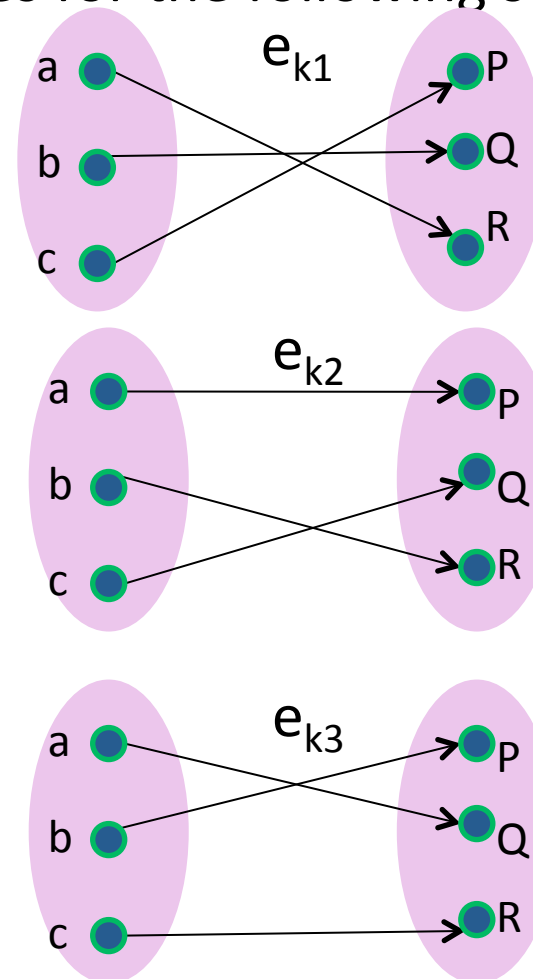
$$\Pr[X=c] = 1/6$$

keyspace

$$\Pr[K=k_1] = 1/3$$

$$\Pr[K=k_2] = 1/3$$

$$\Pr[K=k_3] = 1/3$$



Observations on Perfect Secrecy

Perfect Secrecy iff

Follows from
Baye's theorem

$$\Pr[Y = y \mid X = x] = \Pr[Y = y]$$

Perfect Indistinguishability

$\forall x_1, x_2 \in P$

$$\Pr[Y = y \mid X = x_1] = \Pr[Y = y \mid X = x_2]$$

Perfect secrecy has nothing to do with plaintext distribution.
Thus a crypto-scheme will achieve perfect secrecy irrespective of
the language used in the plaintext.

Shift Cipher with a Twist

- Plaintext set : $P = \{0,1,2,3 \dots, 25\}$
- Ciphertext set : $C = \{0,1,2,3 \dots, 25\}$
- Keyspace : $K = \{0,1,2,3 \dots, 25\}$
- Encryption Rule : $e_K(x) = (x + K) \bmod 26$,
- Decryption Rule : $d_K(x) = (x - K) \bmod 26$
where $K \in K$ and $x \in P$

The Twist :

- (1) the key changes after every encryption
- (2) keys are picked with uniform probability

The Twisted Shift Cipher is Perfectly Secure

$$\Pr[y = y] = \sum_{K \in \mathbb{Z}_{26}} \Pr[K = K] \Pr[x = d_K(y)]$$

Keys chosen with uniform probability

$$= \sum_{K \in \mathbb{Z}_{26}} \frac{1}{26} \Pr[x = y - K]$$

This is 1 because the sum is over all values of x

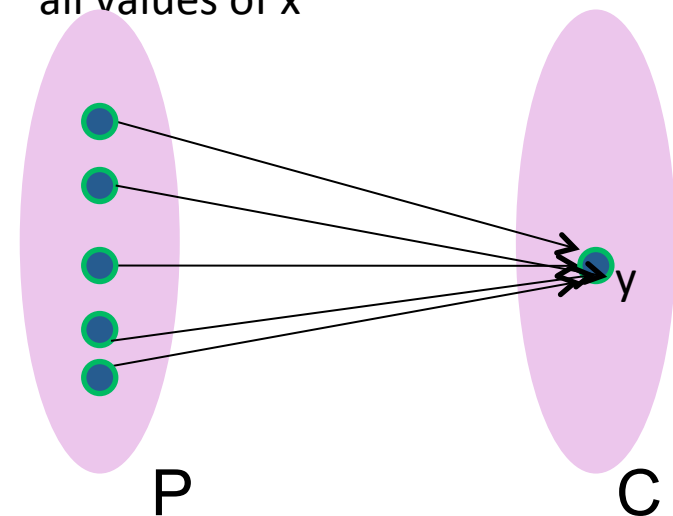
$$= \frac{1}{26} \sum_{K \in \mathbb{Z}_{26}} \Pr[x = y - K]$$

$$= \frac{1}{26}$$

$$\Pr[y|x] = \Pr[K = (y - x) \bmod 26]$$

$$= \frac{1}{26}$$

For every pair of y and x, there is exactly one key. Probability of that key is 1/26



The Twisted Shift Cipher is Perfectly Secure

$$\begin{aligned}\Pr[\mathbf{y} = y] &= \sum_{K \in \mathbb{Z}_{26}} \Pr[\mathbf{K} = K] \Pr[\mathbf{x} = d_K(y)] \\ &= \sum_{K \in \mathbb{Z}_{26}} \frac{1}{26} \Pr[\mathbf{x} = y - K] \\ &= \frac{1}{26} \sum_{K \in \mathbb{Z}_{26}} \Pr[\mathbf{x} = y - K]. \\ &= \frac{1}{26}\end{aligned}$$

$$\begin{aligned}\Pr[x|y] &= \frac{\Pr[x] \Pr[y|x]}{\Pr[y]} \\ &= \frac{\Pr[x] \frac{1}{26}}{\frac{1}{26}} \\ &= \Pr[x],\end{aligned}$$

$$\begin{aligned}\Pr[y|x] &= \Pr[\mathbf{K} = (y - x) \bmod 26] \\ &= \frac{1}{26}\end{aligned}$$

Shannon's Theorem

If $|K| = |C| = |P|$ then the system provides perfect secrecy iff

(1) every key is used with equal probability $1/|K|$, and

(2) for every $x \in P$ and $y \in C$, there exists a unique key $k \in K$ such that $e_k(x) = y$

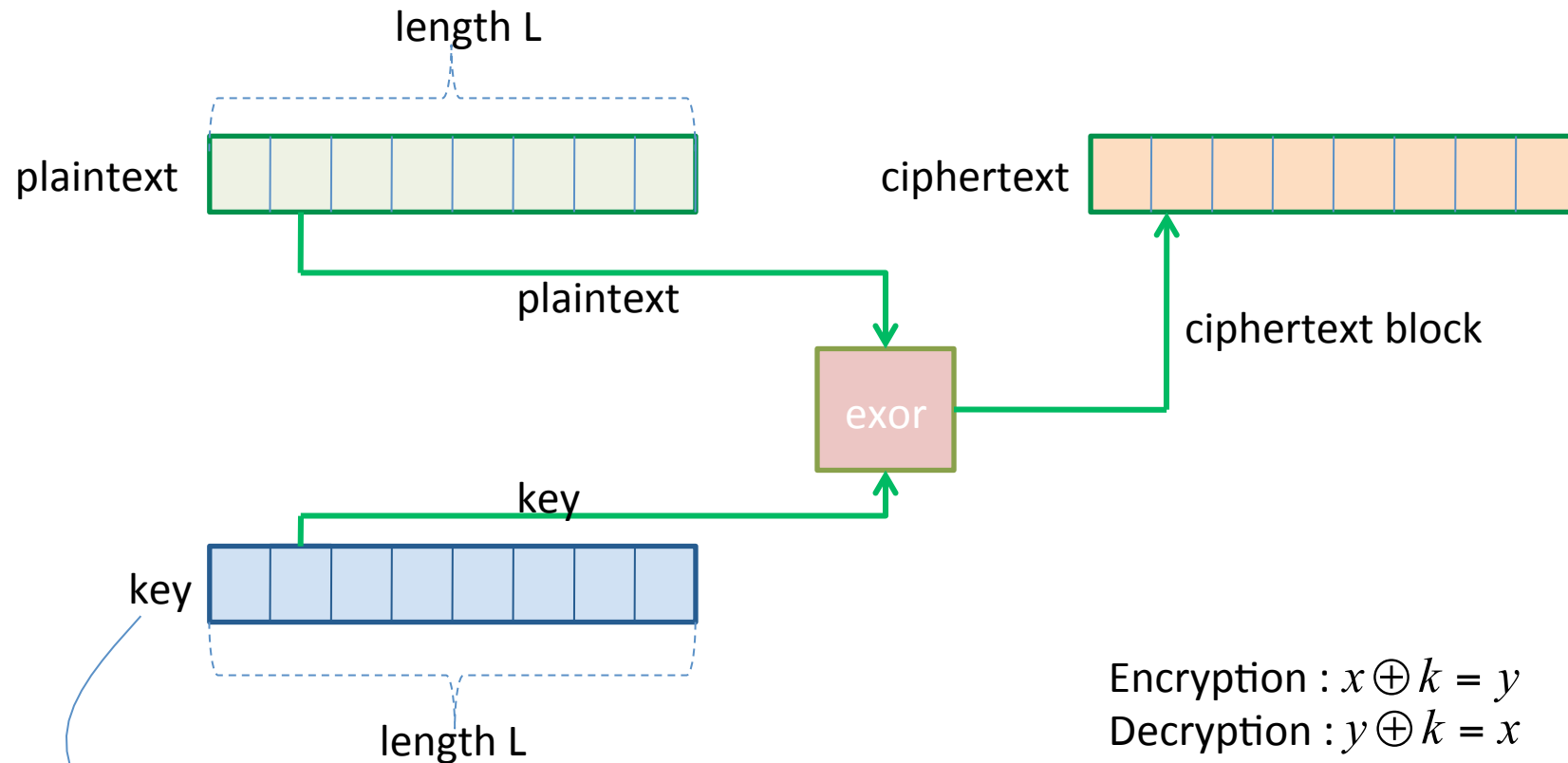
Intuition :

Every $y \in C$ can result from any of the possible plaintexts x

Since $|K| = |P|$ there is exactly one mapping from each plaintext to y

Since each key is equi-probable, each of these mappings is equally probable

One Time Pad (Verman's Cipher)



Encryption : $x \oplus k = y$

Decryption : $y \oplus k = x$

chosen uniformly from keyspace of size 2^L

$\Pr[K = k] = 1/2^L$

One Time Pad (Example)

¹
 e=000 h=001 i=010 k=011 l=100 r=101 s=110 t=111

Encryption: Plaintext \oplus Key = Ciphertext

	h	e	i	l	h	i	t	l	e	r
Plaintext:	001	000	010	100	001	010	111	100	000	101
Key:	111	101	110	101	111	100	000	101	110	000
Ciphertext:	110	101	100	001	110	110	111	001	110	101
	s	r	l	h	s	s	t	h	s	r

One Time Pad is Perfectly Secure

- Proof using indistinguishability

$$\begin{aligned}\Pr[Y = y \mid X = x] &= \Pr[X = x, K = k \mid X = x] \quad \text{from } x \oplus k = y \\ &= \Pr[K = k] = \frac{1}{2^L}\end{aligned}$$

$$\begin{aligned}\Pr[Y = y \mid X = x_1] &= \frac{1}{2^L} = \Pr[Y = y \mid X = x_2] \\ &\quad \forall x_1, x_2 \in X\end{aligned}$$

**This implies perfect Indistinguishability
that is independent of the plaintext distribution**

Limitations of Perfect Secrecy

- Key must be at least as long as the message
 - Limits applicability if messages are long
- Key must be changed for every encryption
 - If the same key is used twice, then an adversary can compute the ex-or of the messages

$$x_1 \oplus k = y_1$$

$$x_2 \oplus k = y_2$$

$$x_1 \oplus x_2 = y_1 \oplus y_2$$

The attacker can then do language analysis to determine y_1 and y_2

Ciphers in Practice

- Perfect secrecy is difficult to achieve in practice
- Computational Security rather than Perfect Security
- Instead we use a crypto-scheme that cannot be *broken in reasonable time* with *reasonable success*
- This means,
 - Security is only achieved against adversaries that run in polynomial time
 - Attackers can potentially succeed with a very small probability (attackers need to be very lucky to succeed)

Quantifying Information

A Metric to Quantify Information

There is one alphabet missing in each of these words. Can you find the alphabet so that the words make sense?

nough
ntwork
dvic

enough
network
device

lassis
hole
lok

classics
chole
clock

Frequently occurring letters (like e) contain less information than non-frequent letters (like c)

We need to have function to quantify information!

Additionally, the function should be (1) continuous (2) should be able to sum individual information (eg. $X1$: Message 1, $X2$: Message 2)

$$I(X1, X2) = I(\text{Message 1}) + I(\text{Message 2})$$

Metric to Quantify Information



Claude Shannon

A higher probability indicates lesser information content.

$$\log_2 \left(\frac{1}{p_i} \right)$$

$$\Pr(e) = 0.12702$$

$$-\log_2(0.12702) = 2.97$$

$$\Pr(a) = 0.08167$$

$$-\log_2(0.08167) = 3.61$$

$$\Pr(m) = 0.02406$$

$$-\log_2(0.02406) = 5.37$$

$$\Pr(c) = 0.02782$$

$$-\log_2(0.02782) = 5.16$$

$$\Pr(q) = 0.0095$$

$$-\log_2(0.0095) = 6.71$$

...

...

...

Metric to Quantify Information



Claude Shannon

To find the average information content of a language
find weighted sum as follows

$$\sum_{i=1}^n p_i \log_2 \left(\frac{1}{p_i} \right)$$

Metric to Quantify Information



Claude Shannon

Entropy provides the average number of bits needed to represent letters in the language

To find the average information content of a language find weighted sum as follows

Call this term the **Entropy**

$$H(X) = \sum_{i=1}^n p_i \log_2 \left(\frac{1}{p_i} \right)$$

Entropy of English

Contemporary : 4.03 bits

Shakespeare : 4.106 bits

German : 4.08 bits

French : 4.00 bits

Italian : 3.98 bits

Spanish : 3.98 bits

Maximum Entropy occurs when each alphabet is equally likely (ie. 1/26).

The maximum entropy is $\log_2(1/26) = 4.7$

Entropy of the Weather Forecast

Weather Forecast

Tomorrow the weather will be

M1 : Sunny (with probability 0.05)

M2 : Cloudy (with probability 0.15)

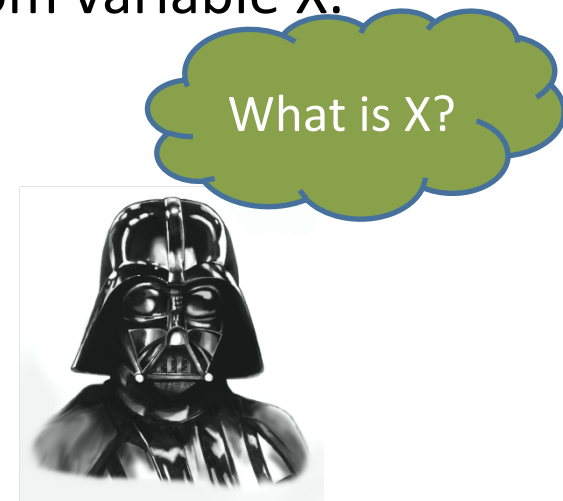
M3 : Light Rain (with probability 0.70)

M4 : Heavy Rain (with probability 0.10)

$$\begin{aligned} H(\text{Forecast}) &= \sum_{i=1}^n p_i \log_2 \left(\frac{1}{p_i} \right) \\ &= -((0.05) \log_2 0.05 + (0.15) \log_2 0.15 + (0.7) \log_2 0.7 + (0.1) \log_2 0.1) \\ &= 1.319 \end{aligned}$$

Entropy and Uncertainty

- Alice thinks of a number (0 or 1)
- The choice is denoted by a discrete random variable X .



- What is Mallory's uncertainty about X ?
 - Depends on the probability distribution of X
(Mallory knows the probability distribution)

Uncertainty

- Lets assume Mallory know this probability distribution.
- If $\Pr[X = 1] = 1$ and $\Pr[X = 0] = 0$
 - Then Mallory can determine with 100% accuracy
- If $\Pr[X = 0] = .75$ and $\Pr[X = 1] = .25$
 - Mallory will guess X as 0, and gets it right 75% of the time
- If $\Pr[X=0] = \Pr[X = 1] = 0.5$
 - Mallory's guess would be similar to a uniformly random guess. Gets it right $\frac{1}{2}$ the time.



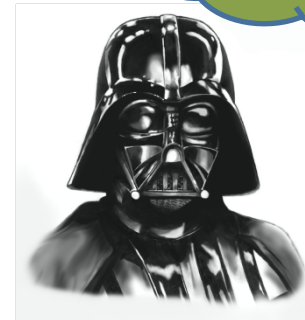
What is X?

What is the Entropy of X?

X



What is X?

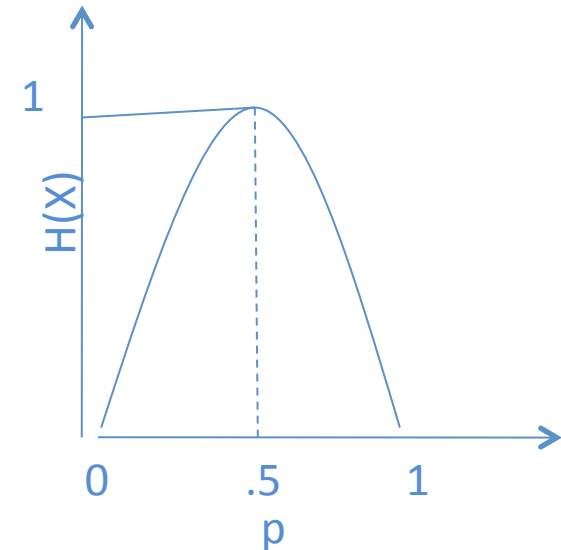


$\Pr[X=0] = p$ and $\Pr[X=1] = 1 - p$

$H(X) = -p \log_2 p - (1-p) \log_2 (1-p)$

$H(X)_{p=0} = 0$, $H(X)_{p=1} = 0$, $H(X)_{p=.5} = 1$

using $\lim_{p \rightarrow 0} (p \log p) = 0$



Properties of $H(X)$

- If X is a random variable, which takes on values $\{1,2,3,\dots,n\}$ with probabilities $p_1, p_2, p_3, \dots, p_n$, then

1. $H(X) \leq \log_2 n$

2. When $p_1 = p_2 = p_3 = \dots p_n = 1/n$ then $H(X) = \log_2 n$

Example an 8 face dice.

If the dice is fair, then we obtain the maximum entropy of 3 bits

If the dice is unfair, then the entropy is < 3 bits

Entropy and Coding

- Entropy quantifies Information content

“Can we encode a message M in such a way that the average length is as short as possible and hopefully equal to $H(M)$?”

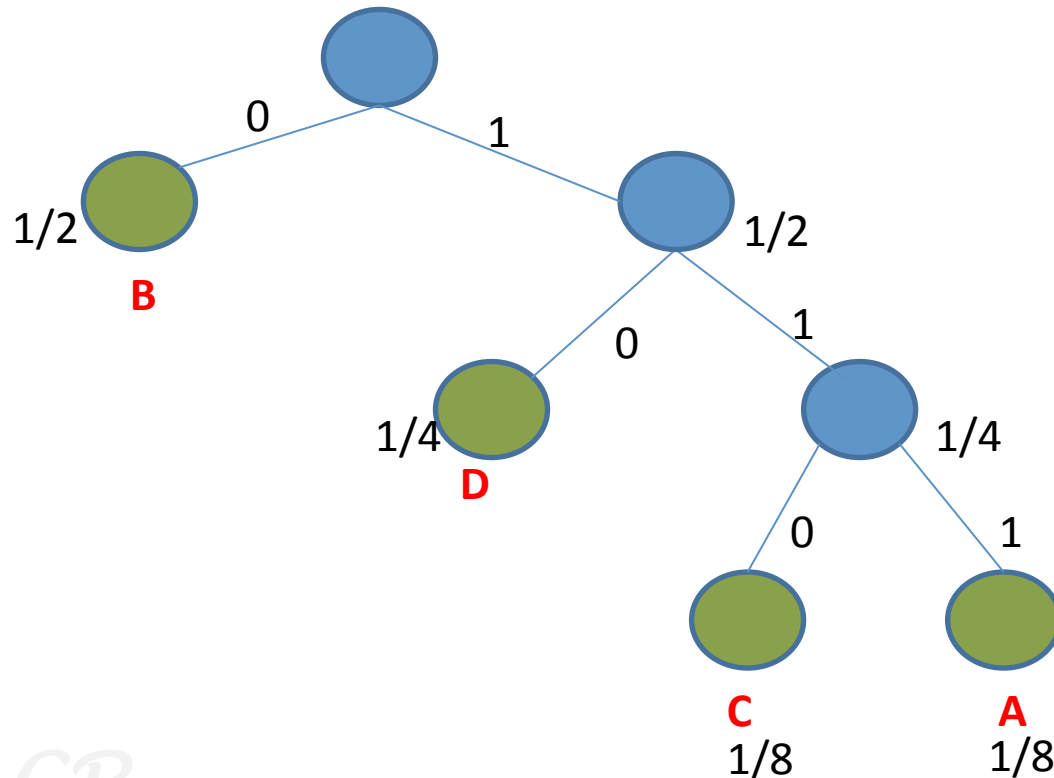
Huffman Codes :

allocate more bits to least probable events
allocate less bits to popular events

Example

- $S = \{A, B, C, D\}$ are 4 symbols
- Probability of Occurrence is :
 $P(A) = 1/8, P(B) = 1/2, P(C) = 1/8, P(D) = 1/4$

Encoding
A : 111
B : 0
C : 110
D : 10



To decode, with each bit
traverse the tree from
root until you reach a
leaf.

Decode this?

1101010111

Example :

Average Length and Entropy

- $S = \{A, B, C, D\}$ are 4 symbols
- Probability of Occurrence is :
 $p(A) = 1/8, p(B) = 1/2, p(C) = 1/8, p(D) = 1/4$
- Average Length of Huffman code :
 $3 * p(A) + 1 * p(B) + 3 * p(C) + 2 * p(D) = 1.75$
- Entropy $H(S) =$
 $-1/8 \log_2(8) - 1/2 \log_2(2) - 1/8 \log_2(8) - 1/4 \log_2(4)$
 $= 1.75$

Encoding
A : 111
B : 0
C : 110
D : 10

Example

Entropy Considering One Letter

- Consider a language with 26 letters of the set $S = \{s_1, s_2, s_3, \dots, s_{26}\}$. Suppose the language is characterized by the following probabilities. **What is the language entropy?**

$$P(s_1) = \frac{1}{2}, P(s_2) = \frac{1}{4}$$

$$P(s_i) = \frac{1}{64} \quad \text{for } i = 3, 4, 5, 6, 7, 8, 9, 10$$

$$P(s_i) = \frac{1}{128} \quad \text{for } i = 11, 12, \dots, 26$$

Maximum Entropy

$$R = \log 26 = 4.7$$

Language Entropy

$$\begin{aligned} r_1 &= H(S^{(1)}) \\ &= \sum_{i=1}^{26} P(s_i) \log \frac{1}{P(s_i)} \\ &= \frac{1}{2} \log 2 + \frac{1}{4} \log 4 + 8 \left(\frac{1}{64} \log 64 \right) + 16 \left(\frac{1}{128} \log 128 \right) \\ &= \frac{1}{2} + \frac{1}{2} + \frac{6}{8} + \frac{7}{8} = 2.625 \end{aligned}$$

Example

Entropy Considering Two Letters

- In the set $S = \{s_1, s_2, s_3, \dots, s_{26}\}$, suppose the diagram probabilities is as below. **What is the entropy?**

$$P(s_{i+1} | s_i) = P(s_{i+2} | s_i) = \frac{1}{2} \quad \text{for } i = 1 \text{ to } 24$$

$$P(s_{26} | s_{25}) = P(s_1 | s_{25}) = P(s_1 | s_{26}) = P(s_2 | s_{26}) = \frac{1}{2}$$

all other probabilities are 0

$$P(s_1, s_2) = P(s_2 | s_1) \times P(s_1) = 1/4; P(s_1, s_3) = P(s_3 | s_1) \times P(s_1) = 1/4$$

$$P(s_2, s_3) = P(s_3 | s_2) \times P(s_2) = 1/8; P(s_2, s_4) = P(s_4 | s_2) \times P(s_2) = 1/8$$

$$P(s_i, s_{i+1}) = P(s_{i+1} | s_i) P(s_i) = 1/128 \quad \text{for } i = 3, 4, \dots, 10$$

$$P(s_i, s_{i+2}) = P(s_{i+2} | s_i) P(s_i) = 1/128 \quad \text{for } i = 3, 4, \dots, 10$$

$$P(s_i, s_{i+1}) = P(s_{i+1} | s_i) P(s_i) = 1/256 \quad \text{for } i = 11, 12, \dots, 24$$

$$P(s_i, s_{i+2}) = P(s_{i+2} | s_i) P(s_i) = 1/256 \quad \text{for } i = 11, 12, \dots, 24$$

$$P(s_{25}, s_{26}) = P(s_{25}, s_1) = P(s_{26}, s_1) = P(s_{26}, s_2) = 1/256$$

Entropy considering 2 letters

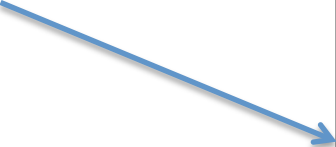
$$\begin{aligned} H(S^{(2)}) &= 2 \sum_{i,j=1}^{26} P(s_i, s_j) \log \frac{1}{P(s_i, s_j)} \\ &= \left[2 \left(\frac{1}{4} \log 4 \right) + 2 \left(\frac{1}{8} \log 8 \right) + 16 \left(\frac{1}{128} \log 128 \right) + 32 \left(\frac{1}{256} \log 256 \right) \right] \\ &= \left[1 + \frac{3}{4} + \frac{7}{8} + 1 \right] = 3.625 \end{aligned}$$

Redundancy in Languages

$$H(S) = 2.625$$

$$H(S^{(2)}) = 3.625$$

$$H(S^{(2)}) - H(S) = 1$$



This means, that having the first letter, we can obtain the second one using one bit only.
i.e. if we know the first letter, then there are only 2 equally possible candidates for the second.

Languages are redundant

Entropy reduces as we consider more number of alphabets in the entropy computation

Measuring the Redundancy in a Language

- Let S be letter in a language (eg. $S = \{A,B,C,D\}$)
- $\mathbf{S} = S \times S \times S \times S \times S \times S$ (k times) is a set representing messages of length k
- Let $S^{(k)}$ be a random variable in \mathbf{S}
- The average information in each letter is given by the **rate of $S^{(k)}$** .

$$r_k = \frac{H(S^{(k)})}{k}$$

In our example,

$$r_1 = H(S) = 2.625$$

$$r_2 = H(S^{(2)}) / 2 = 3.625 / 2 = 1.8125$$

- r_k for English is between 1.0 and 1.5 bits/letter (when k is large)

Measuring the Redundancy in a Language

- **Absolute Rate(R)** : The maximum amount of information per character in a language
 - the absolute rate of language S is $R = \log_2 |S|$
 - For English, $|S| = 26$, therefore $R = 4.7$ bits / letter

- **Redundancy of a language is**

$$D = R - r_k$$

- For English when $r_k = 1$, then $D = 3.7 \rightarrow$ around 70% redundant

$$r_1 = H(S) = 2.625$$

$$D_1 = 4.7 - 2.625 = 2.075 \text{ (44\% redundant)}$$

$$r_2 = H(S^{(2)}) = 1.8125$$

$$D_2 = 4.7 - 1.8125 = 2.8875 \text{ (61\% redundant)}$$

As we increase the message size Rate reduces; inferring less information per letter
Redundancy increases

Conditional Entropy

- Suppose X and Y are two discrete random variables, then conditional entropy is defined as

$$\begin{aligned} H(X | Y) &= \sum_y p(y) \sum_x p(x | y) \log_2 \left(\frac{1}{p(x | y)} \right) \\ &= \sum_x \sum_y p(x, y) \log_2 \left(\frac{p(x)}{p(x, y)} \right) \end{aligned}$$

- Conditional entropy means
 - What is the remaining uncertainty about X given Y
 - $H(X | Y) \leq H(X)$ with equality when X and Y are independent

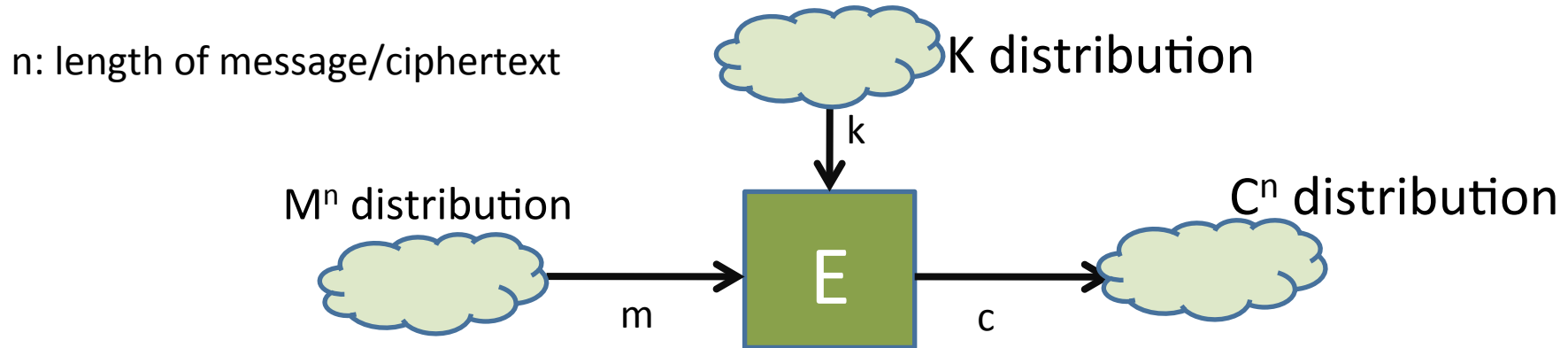
Joint Entropy

- Suppose X and Y are two discrete random variables, and $p(x,y)$ the value of the joint probability distribution when $X=x$ and $Y=y$
- Then the joint entropy is given by

$$H(X,Y) = \sum_y \sum_x p(x,y) \log_2 \left(\frac{1}{p(x,y)} \right)$$

- The joint entropy is the average uncertainty of 2 random variables

Entropy and Encryption



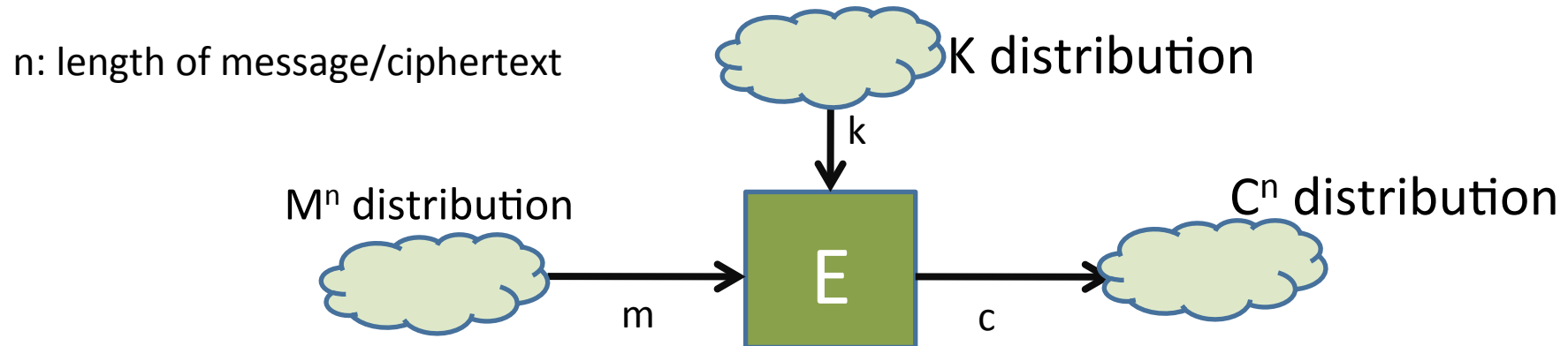
- There are three entropies: $H(P^{(n)})$, $H(K)$, $H(C^{(n)})$

- **Message Equivocation :**

If the attacker can view n ciphertexts, what is his uncertainty about the message

$$H(M^{(n)} | C^{(n)}) = \sum_{c \in C^n} p(c) \sum_{m \in M^n} p(m | c) \log_2 \left(\frac{1}{p(m | c)} \right)$$

Entropy and Encryption



- **Key Equivocation :**

If the attacker can view n ciphertexts, what is his uncertainty about the key

$$H(K | C^{(n)}) = \sum_{c \in C^n} p(c) \sum_{m \in M^n} p(k | c) \log_2 \left(\frac{1}{p(k | c)} \right)$$

Unicity Distance

$$H(K | C^{(n)}) = \sum_{c \in C^n} p(c) \sum_{m \in M^n} p(k | c) \log_2 \left(\frac{1}{p(k | c)} \right)$$

- As n increases, $H(K | C^{(n)})$ reduces...
 - This means that the uncertainty of the key reduces as the attacker observes more ciphertexts
- **Unicity distance** is the value of n for which $H(K | C^{(n)}) \approx 0$
 - This means, the entire key can be determined in this case

Unicity Distance and Classical Ciphers

Cipher	Unicity Distance (for English)
Caesar's Cipher	1.5 letters
Affine Cipher	2.6 letters
Simple Substitution Cipher	27.6 letters
Permutation Cipher	0.12 (block size = 3) 0.66 (block size = 4) 1.32 (block size = 5) 2.05 (block size = 6)
Vigenere Cipher	1.47d (d is the key length)

Product Ciphers

- Consider a cryptosystem where $P=C$ (this is an endomorphic system)
 - Thus the ciphertext and the plaintext set is the same
- Combine two ciphering schemes to build a **product cipher**

Given two endomorphic crypto-systems

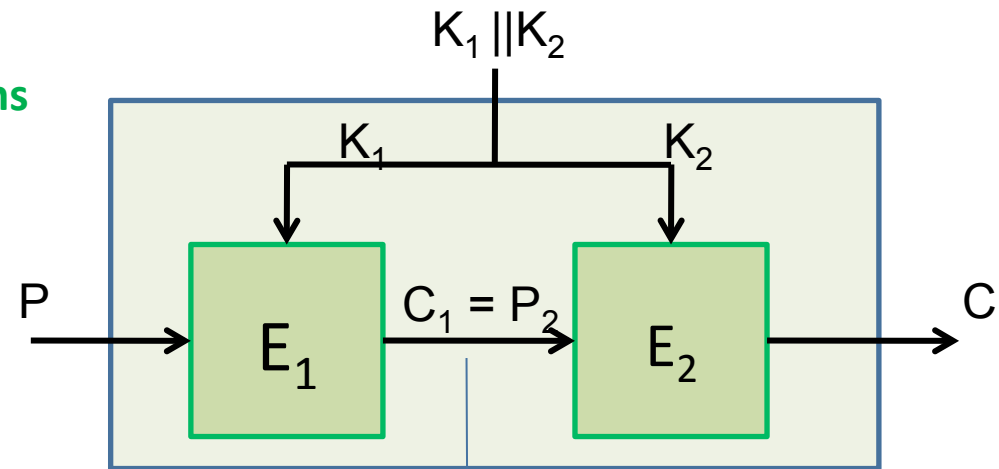
$$S_1 : (P, P, K_1, E_1, D_1)$$

$$S_2 : (P, P, K_2, E_2, D_2)$$

Resultant Product Cipher

$$S_1 \times S_2 : (P, P, K_1 \times K_2, E, D)$$

Resultant Key Space $K_1 \times K_2$



Ciphertext of first cipher fed as input to the second cipher

Product Ciphers

- Consider a cryptosystem where $P=C$ (this is an endomorphic system)
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Given two endomorphic crypto-systems

$$S_1 : x = d_{K_1}(e_{K_1}(x))$$

$$S_2 : x = d_{K_2}(e_{K_2}(x))$$

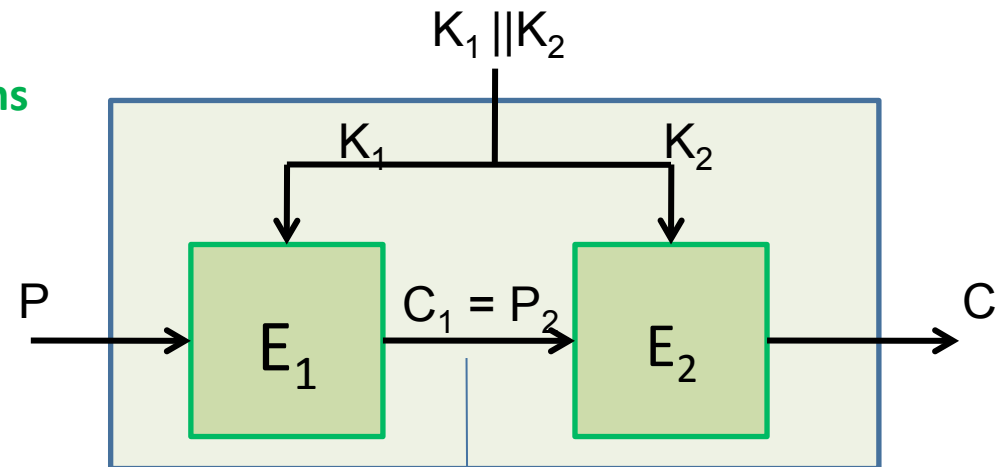
Resultant Product Cipher

$$S_1 \times S_2$$

$$e_{(K_1, K_2)}(x) = e_{K_2}(e_{K_1}(x))$$

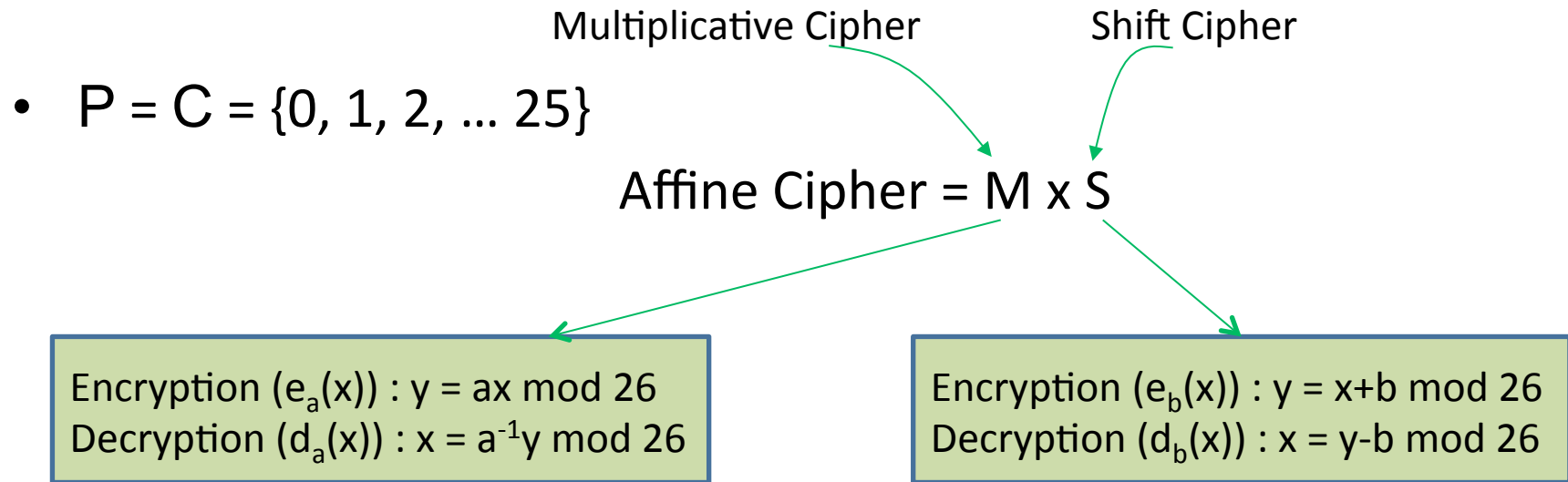
$$d_{(K_1, K_2)}(x) = d_{K_1}(d_{K_2}(x))$$

Resultant Key Space $K_1 \times K_2$



Ciphertext of first cipher fed as input to the second cipher

Affine Cipher is a Product Cipher



- Affine cipher : $y = ax + b \bmod 26$
- Size of Key space is
 - Size of key space for Multiplicative cipher * Size of keyspace for shift cipher
 - $12 * 26 = 312$

Is $S \times M$ same as the Affine Cipher

- $S \times M : y = a(x + b) \bmod 26$
 $= ax + ba \bmod 26$
- Key is (b, a)
- $ba \bmod 26$ is some b' such that
 $a^{-1}b' = b \bmod 26$
- This can be represented as an Affine cipher,
 $y = ax + b' \bmod 26$

Thus affine ciphers are commutable (i.e. $S \times M = M \times S$)

Create a non-commutable product ciphers

Idempotent Ciphers

- If $S_1 : (P, P, K, E_1, D_1)$ is an endomorphic cipher
- then it is possible to construct product ciphers of the form $S_1 \times S_1$, denoted $S^2 : (P, P, K \times K, E, D)$
- If $S^2 = S$ then the cipher is called idempotent cipher

Show that the simple substitution cipher is idempotent

Does the security of the newly formed cipher increase?

In a non-idempotent cipher, however the security may increase.

Iterative Cipher

- An n-fold product of this is $S \times S \times S \dots (n \text{ times}) = S^n$ is an iterative cipher

All modern block ciphers like DES, 3-DES, AES, etc. are iterative, non-idempotent, product ciphers.

We will see more about these ciphers next!!