Reversible Pebble Game on Trees

Balagopal Komarath, Jayalal Sarma, and Saurabh Sawlani

Department of Computer Science & Engineering,
Indian Institute of Technology Madras, Chennai, India.

Abstract. A surprising equivalence between different forms of pebble games on graphs - Dymond-Tompa pebble game (studied in [4]), Raz-McKenzie pebble game (studied in [10]) and reversible pebbling (studied in [1]) - was established recently by Chan[2]. Motivated by this equivalence, we study the reversible pebble game and establish the following results.

- We give a polynomial time algorithm for computing reversible pebbling number of trees. As our main technical contribution, we show that the reversible pebbling number of any tree is exactly one more than the edge rank colouring of the underlying undirected tree.
- By exploiting the connection with the Dymond-Tompa pebble game, we show that complete binary trees have optimal pebblings that take at most $n^{O(\log\log(n))}$ steps. This substantially improves the previous bound of $n^{O(\log(n))}$ steps.
- Furthermore, we show that *almost optimal* (within $(1+\epsilon)$ factor for any constant $\epsilon > 0$) pebblings of complete binary trees can be done in polynomial number of steps.
- We also show a time-space tradeoff for reversible pebbling for families of bounded degree trees: for any constant $\epsilon > 0$, such families can be pebbled using $O(n^{\epsilon})$ pebbles in $O(n)$ steps. This generalizes a result of Královic[7] who showed the same for chains.

1 Introduction

Pebbling games on graphs of various forms abstracts out resources in different combinatorial models of computation (see [3]). The most obvious connection is to the space used by the computation process. A pebble placed on a vertex in a graph corresponds to storing the value at that vertex and an edge (a, b) in the graph would represent a data-dependency - namely, value at b can be computed only if the value at a is known (or stored). Devising the rules of the pebbling game to capture the moves in the computation, and establishing bounds for the total number of pebbles used at any point in time, give rise to a combinatorial approach to proving bounds on the space used by the computation. The Dymond-Tompa pebble game and the Raz-Mckenzie pebble games depict some of the combinatorial barriers in proving bounds for depth (or parallel time) of Boolean circuits (or parallel algorithms).

Motivated by applications in the context of reversible computation (for example, quantum computation), Bennett[1] introduced the reversible pebbling game. Given any DAG G with a unique sink vertex r, the reversible pebbling game starts with no pebbles on G and ends with a pebble (only) on r. Pebbles can be placed or removed from any vertex according to the following two rules.

1. To pebble v, all in-neighbours of v must be pebbled.
2. To unpebble v, all in-neighbours of v must be pebbled.

The goal of the game is to pebble the DAG G using the minimum number of pebbles (also using the minimum number of steps).

Recently, Chan[2] showed that for any DAG G the number of pebbles required for the reversible pebbling game is exactly the same as the number of pebbles required for the Dymond-Tompa pebble game and the Raz-Mckenzie pebble game. Chan[2] also studied the complexity of the following problem – Given a DAG $G = (V, E)$ with sink r and an integer $1 \leq k \leq |V|$, check if G can be pebbled using at most k pebbles. He showed that this problem is PSPACE-complete.

The irreversible black and black-white pebble games are known to be PSPACE-complete on DAGs (see [5], [6]). When we restrict the irreversible black pebbling game to be read-once (each vertex is pebbled only once), then the problem becomes NP-complete (see [11]). However, if we restrict the DAG to a tree, the irreversible black pebble game[9] and black-white pebble game[13] are solvable in polynomial time. The key insight is that optimal *irreversible* (black or black-white) pebbling number of trees can be achieved by read-once pebblings of trees. This fact simplifies many arguments for irreversible pebblings of trees. For example, deciding whether the pebbling number is at most k is in NP since the optimal pebbling can be used as the certificate. We cannot show that reversible pebbling is in NP using the same argument as we do not know whether the optimal value can always be achieved using pebblings taking only polynomially many steps.

Our Results: In this paper, we study reversible pebblings on trees. We show that the reversible pebbling number of trees along with strategies achieving the optimal value can be computed in polynomial time. Our main technical result is that the reversible pebbling number of any tree is exactly one more than the edge rank colouring of the underlying undirected tree. We then use the linear-time algorithm given by Lam and Yue [8] for finding an optimal edge rank coloring of the underlying undirected tree and show how to convert an optimal edge rank coloring into an optimal reversible pebbling.

Chan[2] also raised the question whether we can find connections between other parameters of different pebbling games. Although, we do not answer this question, we show that the connection with Dymond-Tompa pebble game can be exploited to show that complete binary trees have optimal pebblings that take at most $n^{O(\log\log(n))}$ steps. This is a significant improvement over the trivial $n^{O(\log(n))}$ steps.

Furthermore, we show that "almost" (within $(1 + \epsilon)$ factor for any constant $\epsilon > 0$) optimal pebblings of complete binary trees can be done in polynomial number of steps. We also generalize a time-space tradeoff result given for chains by Královic [7] to families of bounded degree trees showing that for any constant $\epsilon > 0$, such families can be pebbled using $O(n^{\epsilon})$ pebbles in $O(n)$ steps.

2 Preliminaries

We assume familiarity with basic definitions in graph theory, such as those found in [12]. A directed tree $T = (V, E)$ is called a *rooted directed tree* if there is an $r \in V$ such that r is reachable from every vertex in T. The vertex r is called the root of the tree.

An *edge rank coloring* of an undirected tree T with k colours $\{1, \ldots, k\}$ labels each edge of T with a colour such that if two edges have the same colour i, then the path between these two edges consists of an edge with some colour $j > i$. The minimum number of colours required for an edge rank colouring of T is denoted by $\chi'_e(T)$.

Definition 1. *(Reversible Pebbling[1]) Let G be a rooted DAG with root r. A* reversible pebbling configuration *of G is a subset of V which denotes the set of pebbled vertices). A* reversible pebbling *of G is a sequence of reversible pebbling configurations $\mathcal{P} = (P_1, \ldots, P_m)$ such that $P_1 = \phi$ and $P_m = \{r\}$ and for every $i, 2 \leq i \leq m$, we have*

1. *$P_i = P_{i-1} \cup \{v\}$ or $P_{i-1} = P_i \cup \{v\}$ and $P_i \neq P_{i-1}$ (Exactly one vertex is pebbled/unpebbled at each step).*
2. *All in-neighbours of v are in P_{i-1}.*

The number m is called the time taken by the pebbling $\mathcal{P}$. The number of pebbles or space used in a reversible pebbling of G is the maximum number of pebbles on G at any time during the pebbling. The persistent reversible pebbling number *of G, denoted by $R^{\bullet}(G)$, is the minimum number of pebbles required to pebble G.*

A closely related notion is that of visiting *reversible pebbling, where the pebbling $\mathcal{P}$ satisfies (1) $P_1 = P_m = \phi$ and (2) there exists a j such that $r \in P_j$. The minimum number of pebbles required for a visiting pebbling of G is denoted by $R^{\phi}(T)$.*

It is easy to see that $R^{\phi}(G) \leq R^{\bullet}(G) \leq R^{\phi}(G) + 1$ for any DAG G.

Definition 2. *(Dymond-Tompa Pebble Game [4]) Let G be a DAG with root r. A Dymond-Tompa pebble game is a two-player game on G where the two players, the pebbler and the challenger take turns. In the first round, the pebbler pebbles the root vertex and the challenger challenges the root vertex. In each subsequent round, the pebbler pebbles a (unpebbled) vertex in G and the challenger either challenges the vertex just pebbled or re-challenges the vertex challenged in the previous round. The pebbler wins when the challenger challenges a vertex v and all in-neighbours of v are pebbled.*

The Dymond-Tompa pebble number of G, denoted $DT(G)$, is the minimum number of pebbles required by the pebbler to win against an optimal challenger play.

The Raz-Mckenzie pebble game is also a two-player pebble game played on DAGs. The optimal value is denoted by $RM(G)$. A definition for the Raz-Mckenzie pebble game can be found in [10]. Although the Dymond-Tompa game and the reversible pebbling game look quite different. The following theorem reveals a surprising connection between them.

Theorem 1. *(Theorems 6 and 7, [2]) For any rooted DAG G, we have $DT(G) = R^{\bullet}(G) = RM(G)$.*

Definition 3. *(Effective Predecessor [2]) Given a pebbling configuration P of a DAG G with root r, a vertex v in G is called an* effective predecessor *of r if there exists a path from v to r with no pebbles on the vertices in the path (except at r).*

Lemma 1. *(Claim 3.11, [2]) Let G be any rooted DAG. There exists an optimal pebbler strategy for the Dymond-Tompa pebble game on G such that the pebbler always pebbles an effective predecessor of the currently challenged vertex.*

We call the above pebbling strategy (resp. pebbler) as an upstream pebbling strategy(resp. upstream pebbler). The height or depth of a tree is defined as the maximum number of vertices in any root to leaf path. We denote by Ch_n the rooted directed path on n vertices with a leaf as the root. We denote by Bt_h the the complete binary tree of height h. We use $root(Bt_h)$ to refer to the root of Bt_h. If v is any vertex in Bt_h, we use $left(v)$ ($right(v)$) to refer to the left (right) child of v. We use $right^i$ and $left^i$ to refer to iterated application of these functions. We use the notation $Ch_i + Bt_h$ to refer to a tree that is a chain of i vertices where the source vertex is the root of a Bt_h.

Definition 4. *We define the language* TREE-PEBBLE *(*TREE-VISITING-PEBBLE*) as the set of all tuples (T, k), where T is a rooted directed tree and k is an integer satisfying $1 \leq k \leq n$, such that $R^{\bullet}(T) \leq k$ ($R^{\phi}(T) \leq k$).*

In the rest of the paper, we use the term pebbling to refer to *persistent reversible pebbling* unless explicitly stated otherwise.

3 Main Theorem

Definition 5. *(Strategy Tree) Let T be a rooted directed tree. If T only has a single vertex v, then any strategy tree for T only has a single vertex labelled v. Otherwise, we define a strategy tree for T as any tree satisfying*

1. *The root vertex is labelled with some edge $e = (u, v)$ in T.*
2. *The left subtree of root is a strategy tree for T_u and the right subtree is a strategy tree for $T \setminus T_u$.*

The following properties are satisfied by any strategy tree S of $T = (V, E)$.

1. Each vertex has 0 or 2 children.
2. There are bijections from E to internal vertices of S & from V to leaves of S.
3. Let v be any vertex in S. Then the subtree S_v corresponds to the subtree of T spanned by the vertices labelling the leaves of S_v. If u and v are two vertices in S such that one is not an ancestor of the other, then the subtrees in T corresponding to u and v are vertex-disjoint.

Lemma 2. *Let T be a rooted directed tree. Then $R^\bullet(T) \leq k$ if and only if there exists a strategy tree for T of depth at most k.*

Proof. We prove both directions by induction on $|T|$. If T is a single vertex tree, then the statement is trivial.

(if) Assume that the root of a strategy tree for T of depth k is labelled by an edge (u, v) in T. The pebbler then pebbles the vertex u. If the challenger challenges u, the pebbler follows the strategy for T_u given by the left subtree of root. If the challenger rechallenges, the pebbler follows the strategy for $T \setminus T_u$ given by the right subtree of the root. The remaining game takes at most $k-1$ pebbles by the inductive hypothesis. Therefore, the total number of pebbles used is at most k.

(only if) Consider an upstream pebbler that uses at most k pebbles. We are going to construct a strategy tree of depth at most k. Assume that the pebbler pebbles u in the first move where $e = (u, v)$ is an edge in T. Then the root vertex of S is labelled e. Now we have $R^\bullet(T_u), R^\bullet(T \setminus T_u) \leq k-1$. Let the left (right) subtree be the strategy tree obtained inductively for T_u ($T \setminus T_u$). Since the pebbler is upstream, the pebbler never places a pebble outside T_u ($T \setminus T_u$) once the challenger has challenged u (the root). $\square$

Definition 6. *(Matching Game) Let U be an undirected tree. Let $T_1 = U$. At each step of the matching game, we pick a matching M_i from T_i and contract all the edges in M_i to obtain the tree T_{i+1}. The game ends when T_i is a single vertex tree. We define the* contraction number *of U, denoted $c(U)$, as the minimum number of matchings in the matching sequence required to contract U to the single vertex tree.*

Lemma 3. *Let T be a rooted directed tree and let U be the underlying undirected tree for T. Then $R^\bullet(T) = k+1$ if and only if $c(U) = k$.*

Proof. First, we describe how to construct a matching sequence of length k from a strategy tree S of depth $k+1$. Let the leaves of S be the level 0 vertices. For $i \geq 1$, we define the level i vertices to be the set of all vertices v in S such that one child of v has level $i-1$ and the other child of v has level at most $i-1$. Define M_i to be the set of all edges in U corresponding to level i vertices in S. We claim that $M_1, \ldots, M_k$ is a matching sequence for U. Define S_i as the set of all vertices v in S such that the parent of v has level at least $i+1$ (S_k contains only the root vertex). Let $Q(i)$ be the statement "T_{i+1} is obtained from T_1 by contracting all subtrees corresponding to vertices (see Property 3) in S_i". Let

$P(i)$ be the statement "M_{i+1} is a matching in T_{i+1}". We will prove $Q(0)$ and $Q(i) \implies P(i)$ and $(Q(i) \wedge P(i)) \implies Q(i+1)$. Indeed for $i=0$, we have $Q(0)$ because $T_1 = U$ and S_0 is the set of all leaves in S or vertices in T (Property 2). To prove $Q(i) \implies P(i)$, observe that the edges of M_{i+1} correspond to vertices in S where both children are in S_i. So these edges correspond to edges in T_{i+1} (by $Q(i)$) and these edges are pairwise disjoint since no two vertices in S have a common child.

To prove that $(Q(i) \wedge P(i)) \implies Q(i+1)$, consider the tree T_{i+2} obtained by contracting M_{i+1} from T_{i+1}. Since $Q(i)$ is true, this is equivalent to contracting all subtrees corresponding to S_i and then contracting the edges in M_{i+1} from T_1. The set S_{i+1} can be obtained from S_i by adding all vertices in S corresponding to edges in M_{i+1} and then removing both children (of these newly added vertices) from S_i. This is equivalent to combining the subtrees removed from S_i using the edge joining them. This is because M_{i+1} is a matching by $P(i)$ and hence one subtree in S_i will never be combined with two other subtrees in S_i. But then contracting subtrees in S_{i+1} from T_1 is equivalent to contracting S_i followed by contracting M_{i+1}.

We now show that a matching sequence of length at most k can be converted to a strategy tree of depth at most $k+1$. We use proof by induction. If the tree T is a single vertex tree, then the statement is trivial. Otherwise, let e be the edge in the last matching M_k in the sequence and let (u, v) be the corresponding edge in T. Label the root of S by e and let the left (right) subtree of root of S be obtained from the matching sequence $M_1, \ldots, M_{k-1}$ restricted to T_u ($T \setminus T_u$). By the inductive hypothesis, these subtrees have height at most $k-1$. □

Lemma 4. *For any undirected tree U, we have $c(U) = \chi'_e(U)$.*

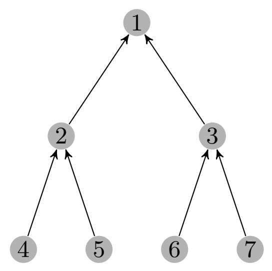
Proof. Consider an optimal matching sequence for U. If the edge e is contracted in M_i, then label e with the color i. This is an edge rank coloring. Suppose for contradiction that there exists two edges e_1 and e_2 with label i such that there is no edge labelled some $j \geq i$ between them. We can assume without loss of generality that there is no edge labelled i between e_1 and e_2 since if there is one such edge, we can let e_2 to be that edge. Then e_1 and e_2 are adjacent in T_i and hence cannot belong to the same matching.

Consider an optimal edge rank coloring for U. Then in the i^{th} step all edges labelled i are contracted. This forms a matching since in between any two edges labelled i, there is an edge labelled $j > i$ and hence they are not adjacent in T_i. □

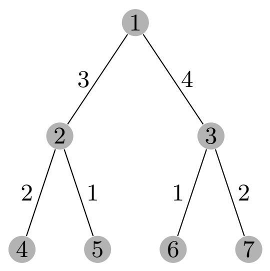
The theorems in this section are summarized in Fig. 1

Theorem 2. *Let T be a rooted directed tree and let U be the underlying undirected tree for T. Then we have $R^\bullet(T) = \chi'_e(U) + 1$.*

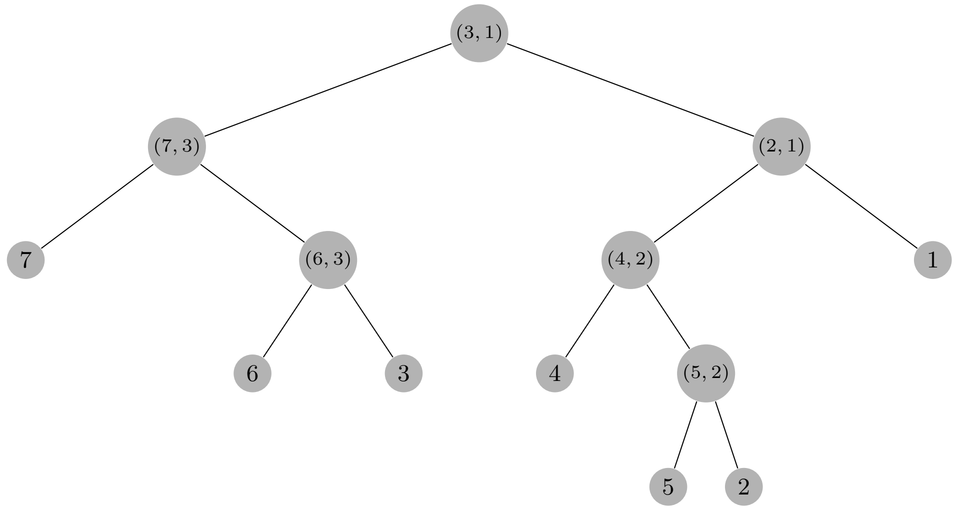
Corollary 1. *$R^\phi(T)$ and $R^\bullet(T)$ along with strategy trees achieving the optimal pebbling value can be computed in polynomial time for trees.*



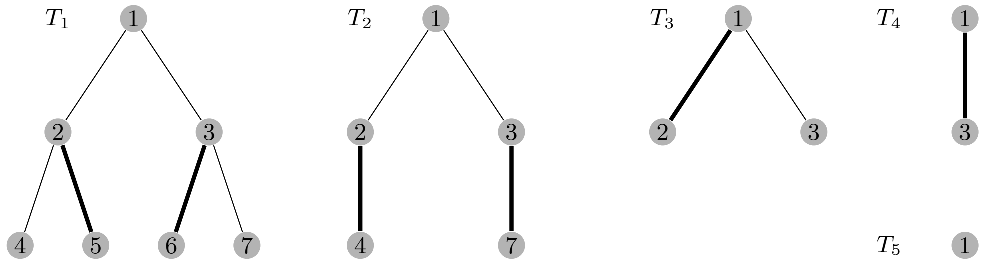
(a) The complete binary tree of height 3



(b) Optimal edge rank colouring



(c) Optimal strategy tree



(d) Optimal matching sequence

Fig. 1: This figure illustrates the equivalence between persistent reversible pebbling, matching game and edge rank coloring on trees by showing an optimal strategy tree and the corresponding matching sequence and edge rank colouring for height 3 complete binary tree.

Proof. We show that TREE-PEBBLE and TREE-VISITING-PEBBLE are polynomial time equivalent. Let T be an instance of TREE-PEBBLE. Pick an arbitrary leaf v of T and root the tree at v. By Theorem 2, the reversible pebbling number of this tree is the same as that of T. Let T' be the subtree rooted at the child of v. Then we have $R^{\bullet}(T) \leq k \iff R^{\phi}(T') \leq k-1$.

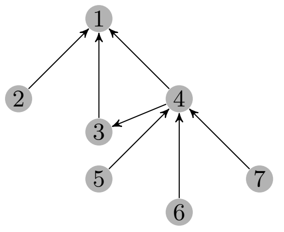
Let T be an instance of TREE-VISITING-PEBBLE. Let T' be the tree obtained by adding the edge (r, r') to T where r is the root of T. Then we have $R^{\phi}(T) \leq k \iff R^{\bullet}(T') \leq k+1$.

The statement of the theorem follows from Theorem 2 and the linear-time algorithm for finding an optimal edge rank coloring of trees[8]. ◻

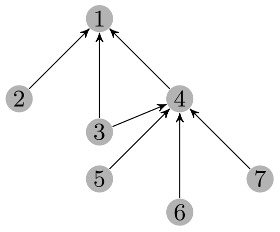
The following corollary is immediate from Theorem 1.

Corollary 2. *For any rooted directed tree T, we can compute $DT(T)$ and $RM(T)$ in polynomial time.*

An interesting consequence of Theorem 2 is that the persistent reversible pebbling number of a tree depends only on its underlying undirected graph. We remark that this does not generalize to DAGs. Below we show two DAGs with the same underlying undirected graph and different pebbling numbers.



(a) $R^{\bullet}(G_1) = 5$



(b) $R^{\bullet}(G_2) = 6$

Fig. 2: DAGs G_1 and G_2 have the same underlying undirected graph and different persistent pebbling numbers.

4 Time Upper-bound for an Optimal Pebbling of Complete Binary Trees

Proposition 1. *The following statements hold.*

1. $R^{\bullet}(Bt_h) \geq R^{\bullet}(Bt_{h-1}) + 1$
2. $R^{\bullet}(Bt_h) \geq h + 2$ *for* $h \geq 3$
3. *([1])* $R^{\bullet}(Ch_n) \leq \lceil \log_2(n) \rceil + 1$ *for all* n

Proof. (1) In any persistent pebbling of Bt_h, consider the earliest time after pebbling the root at which one of the subtrees of the root vertex has $R^{\phi}(Bt_{h-1})$ pebbles. At this time, there is a pebble on the root and there is at least one pebble on the other subtree of the root vertex. So, in total, there are at least $R^{\phi}(Bt_{h-1}) + 2 \geq R^{\bullet}(Bt_{h-1}) + 1$ pebbles on the tree.

(2) Item (1) and the fact that $R^{\bullet}(Bt_3) = 5$. □

Theorem 3. *There exists an optimal pebbling of Bt_h that takes at most $n^{O(\log\log(n))}$ steps.*

Proof. We will describe an optimal upstream pebbler in a pebbler-challenger game who pebbles $root(Bt_h)$, $left(root(Bt_h))$, $left(right(root(Bt_h)))$ and so on. In general, the pebbler pebbles $left(right^{i-1}(root(Bt_h)))$ in the i^{th} step for $1 \leq i < h - \log(h)$. An upper bound on the number of steps taken (denoted by $t(h)$) by the reversible pebbling obtained from this game (which is, recursively pebble $left(right^{i-1}(root(Bt_h)))$ for $0 \leq i < h - \log(h)$ and optimally pebble the remaining tree $Ch_{h-\log(h)} + Bt_{\log(h)}$ using any algorithm) is given below. Here the term $(2h - \log(h) + 1)^{3\log(h)}$ is an upper bound on the number of different pebbling configurations with $3\log(h)$ pebbles, and therefore an upper bound for time taken for optimally pebbling the tree $Ch_{h-\log(h)} + Bt_{\log(h)}$.

$$\begin{aligned} t(h) &\leq 2\left[t(h-1) + t(h-2) + \ldots + t(\log(h)+1)\right] + (2h - \log(h) + 1)^{3\log(h)} \\ &\leq 2ht(h-1) + (2h - \log(h) + 1)^{3\log(h)} \\ &= O\left((2h)^h (2h)^{3\log(h)}\right) \\ &= (\log(n))^{O(\log(n))} = n^{O(\log\log(n))} \end{aligned}$$

In the first step, the pebbler will place a pebble on $left(root(Bt_h))$ and the challenger will re-challenge the root vertex. These moves are optimal. Before the i^{th} step, the tree has pebbles on the root and $left(right^j(root(Bt_h)))$ for $0 \leq j < i-1$. We argue that if $i < h - \log(h)$, placing a pebble on $left(right^{i-1}(root(Bt_h)))$ is an optimal move. If the pebbler makes this move, then the cost of the game is $\max(R^{\bullet}(Bt_{h_1-1}), R^{\bullet}(Ch_i + Bt_{h_1-1})) = R^{\bullet}(Ch_i + Bt_{h_1-1}) \leq R^{\bullet}(Bt_{h_1-1}) + 1 = p$, where $h_1 = h - i + 1$. Note that the inequality here is true when $i < h - \log(h)$ by Prop 1. We consider all other possible pebble placements on i^{th} step and prove that all of them are inferior.

- *A pebble is placed on the path from the root to $right^{i-1}(root(Bt_h))$ (inclusive)*: The challenger will challenge the vertex on which this pebble is placed. The cost of this game is then at least $R^{\bullet}(Bt_{h_1}) \geq p$.
- *A pebble is placed on a vertex with height less than $h_1 - 1$*: The challenger will re-challenge the root vertex and the cost of the game is at least $R^{\bullet}(Ch_i + Bt_{h_1-1})$.

The theorem follows. □

5 Almost Optimal Pebblings of Complete Binary Trees

In this section, we show that we can get arbitrarily close to optimal pebblings for complete binary trees using a polynomial number of steps.

Theorem 4. *For any constant $\epsilon > 0$, we can pebble Bt_h using at most $(1+\epsilon)h$ pebbles and $n^{O(\log(1/\epsilon))}$ steps for sufficiently large h.*

Proof. Let $k \geq 1$ be an integer. Then consider the following pebbling strategy parameterized by k.

1. Recursively pebble the subtrees rooted at $left(right^i(root(Bt_h)))$ for $0 \leq i \leq k-1$ and $right^k(root(Bt_h))$.
2. Leaving the $(k+1)$ pebbles on the tree (from the previous step), pebble the root vertex using an additional k pebbles in $2k-1$ steps.
3. Retaining the pebble on the root, reverse step (1) to remove every other pebble from the tree.

The number of pebbles and the number of steps used by the above strategy on Bt_h for sufficiently large h is given by the following recurrences.

$$S(h) \leq S(h-k) + (k+1) \leq \frac{(k+1)}{k}h$$

$$T(h) \leq 2\left[\sum_{i=1}^{k} T(h-i)\right] + (2k+2) \leq (2k)^h(2k+2) \leq n^{\log(k)+1}(2k+2)$$

where n is the number of vertices in Bt_h.

If we choose $k > 1/\epsilon$, then the theorem follows. □

6 Time-space Trade-offs for Bounded-degree Trees

In this section, we study time-space trade-offs for bounded-degree trees.

Theorem 5. *For any constant positive integer k, a bounded-degree tree T consisting of n vertices can be pebbled using at most $O\left(n^{1/k}\right)$ pebbles and $O\left(2^k n\right)$ pebbling moves.*

Proof. Let us prove this by induction on the value of k. In the base case ($k = 1$), we are allowed to use $O(n)$ pebbles. So, the best strategy would to place a pebble on every vertex of T in bottom-up fashion, starting from the leaf vertices. After the root is pebbled, we unpebble each vertex in exactly the reverse order, while leaving the root pebbled.

In this strategy, clearly, each vertex is pebbled and unpebbled at most once. Hence the number of pebbling moves must be bounded by $2n$. Hence, a tree can be pebbled using $O(n)$ pebbles in $O(2n)$ moves.

Now consider that for $k \leq k_0 - 1$, where k_0 is an integer ≥ 2, any bounded-degree tree T with n vertices can be pebbled using $O\left(n^{1/k}\right)$ pebbles in $O\left(2^k n\right)$ moves. Assume that we are allowed $O\left(n^{1/k_0}\right)$ pebbles. To apply induction, we will be decomposing the tree into smaller components. We prove the following.

Claim. Let T' be any bounded-degree tree with $n' > n^{(k_0-1)/k_0}$ vertices and maximum degree Δ. There exists a subtree T'' of T' such that the number of vertices in T'' is at least $\lfloor n^{(k_0-1)/k_0}/2 \rfloor$ and at most $\lceil n^{(k_0-1)/k_0} \rceil$.

Proof. From the classical tree-separator theorem, we know that T' can be divided into two subtrees, where the larger subtree has between $\lfloor n'/2 \rfloor$ and $\left\lceil n' \cdot \frac{\Delta}{\Delta+1} \right\rceil$ vertices. The key is to recursively subdivide the tree in this way and continually choose the larger subtree. However, we need to show that in doing this we will definitely strike upon a subtree with the number of vertices within the required range. Let $T'_1, T'_2, \ldots$ be the sequence of subtrees we obtain in these iterations. Also let n_i be the number of vertices in T'_i for every i. Note that $\forall i, \lfloor n_i/2 \rfloor \leq n_{i+1} \leq \left\lceil v_i \cdot \frac{\Delta}{\Delta+1} \right\rceil$. Assume that j is the last iteration where $n_j > \lceil n^{(k_0-1)/k_0} \rceil$. Clearly $n_{j+1} \geq \lfloor n^{(k_0-1)/k_0}/2 \rfloor$. Also, by the definition of j, $n_{j+1} \leq \lceil n^{(k_0-1)/k_0} \rceil$. Hence the proof. □

The final strategy will be as follows:

1. Separate the tree into $\theta(n^{1/k_0})$ connected subtrees, each containing $\theta(n^{(k_0-1)/k_0})$ vertices. Claim 6 shows that this can always be done.
2. Let us number these subtrees in the following inductive fashion: denote by T_1, the 'lowermost' subtree, i.e. every path to the root of T_1 must originate from a leaf of T. Denote by T_i, the subtree for which every path to the root originates from either a leaf of T or the root of some T_j for $j < i$. Also, let n_i denote the number of vertices in T_i.
3. Pebble T_1 using $O\left(n_1^{1/(k_0-1)}\right) = O\left(n^{1/k_0}\right)$ pebbles. From the induction hypothesis, we know that this can be done using $O\left(2^{k_0-1} n_1\right)$ pebbling moves.
4. Retaining the pebble on the root vertex of T_1, proceed to pebble T_2 in the same way as above. Continue this procedure till the root vertex of T is pebbled. Then proceed to unpebble every vertex other than the root of T by executing every pebble move upto this instant in reverse order.

Now we argue the bounds on the number of pebbles and pebbling moves of the algorithm. Recall that the number of these subtrees is $O\left(n^{1/k_0}\right)$. Therefore, the number of intermediate pebbles at the root vertices of these subtrees is $O\left(n^{1/k_0}\right)$. Additionally, while pebbling the last subtree, $O\left(n^{1/k_0}\right)$ pebbles are used. Therefore, the total number of pebbles at any time remains $O\left(n^{1/k_0}\right)$. Each of the subtrees are pebbled and unpebbled once (effectively pebbled twice). Thus, the total number of pebbling moves is at most $\sum_i 2O\left(2^{k_0-1} n_i\right) = O\left(2^{k_0} n\right)$. □

7 Discussion & Open Problems

We studied reversible pebbling on trees. Although there are polynomial time algorithms for computing black and black-white pebbling numbers for trees, it was unclear, prior to our work, whether the reversible pebbling number for trees could be computed in polynomial time. We also established that almost optimal pebbling can be done in polynomial time.

We conclude with the following open problems.

- Prove or disprove that there is an optimal pebbling for complete binary trees that takes at most $O\left(n^k\right)$ steps for a fixed k.
- Prove or disprove that the there is a constant k such that optimal pebbling for any tree takes at most $O\left(n^k\right)$ (for black and black-white pebble games, this statement is true with $k = 1$).
- Give a polynomial time algorithm for computing optimal pebblings of trees that take the smallest number of steps.

References

1. Charles H. Bennett. Time/space trade-offs for reversible computation. *SIAM Journal of Computing*, 18(4):766–776, August 1989.
2. Siu Man Chan. Just a pebble game. In *Proceedings of the 28th Conference on Computational Complexity, (CCC)*, pages 133–143, 2013.
3. Siu Man Chan. *Pebble Games and Complexity*. PhD thesis, EECS Department, University of California, Berkeley, Aug 2013.
4. Patrick W. Dymond and Martin Tompa. Speedups of deterministic machines by synchronous parallel machines. *Journal of Computer and System Sciences*, 30(2):149 – 161, 1985.
5. John R. Gilbert, Thomas Lengauer, and Robert Endre Tarjan. The pebbling problem is complete in polynomial space. *SIAM Journal on Computing*, 9(3):513–524, 1980.
6. Philipp Hertel and Toniann Pitassi. The pspace-completeness of black-white pebbling. *SIAM J. Comput.*, 39(6):2622–2682, April 2010.
7. Richard Královic. Time and space complexity of reversible pebbling. In Leszek Pacholski and Peter Ruzicka, editors, *SOFSEM 2001: Theory and Practice of Informatics*, volume 2234 of *Lecture Notes in Computer Science*, pages 292–303. 2001.
8. Tak Wah Lam and Fung Ling Yue. Optimal edge ranking of trees in linear time. In *Proc. of the 9th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 436–445, 1998.
9. Michael C Loui. The space complexity of two pebbles games on trees. Technical Report MIT/LCS/TM-133, Massachusetts Institute of Technology, 1979.
10. Ran Raz and Pierre McKenzie. Separation of the monotone NC hierarchy. *Combinatorica*, 19(3):403–435, 1999. Conference version appeared in proceedings of 38th Annual Symposium on Foundations of Computer Science (FOCS 1997, Pages 234–243).
11. Ravi Sethi. Complete register allocation problems. *SIAM Journal on Computing*, pages 226–248, 1975.
12. Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 2 edition, September 2000.
13. Mihalis Yannakakis. A polynomial algorithm for the min-cut linear arrangement of trees. *Journal of the ACM*, 32(4):950–988, October 1985.