Complexity of Matrix Rank and Rigidity †

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Matrix Rank

Rank of a matrix $M \in \mathbb{F}^{n \times n}$ has the following equivalent definitions.

- The size of the largest submatrix with a non-zero determinant.
- The number of linearly independent rows/columns of a matrix.

RANK BOUND: Given a matrix M and a value r, is rank$(M) < r$?.

Motivation

- **From Linear Algebra** : Dimension of solution space of a system of linear equations.
- **From Control Theory** : Rank of a matrix can be used to classify a linear system as controllable, or observable.
- **From Algorithmics** : Some natural algorithmic problems can be expressed in terms of rank and determinant computation.
- **From Complexity Theory** : Characterising complexity classes might facilitate application of the well studied algebraic techniques.

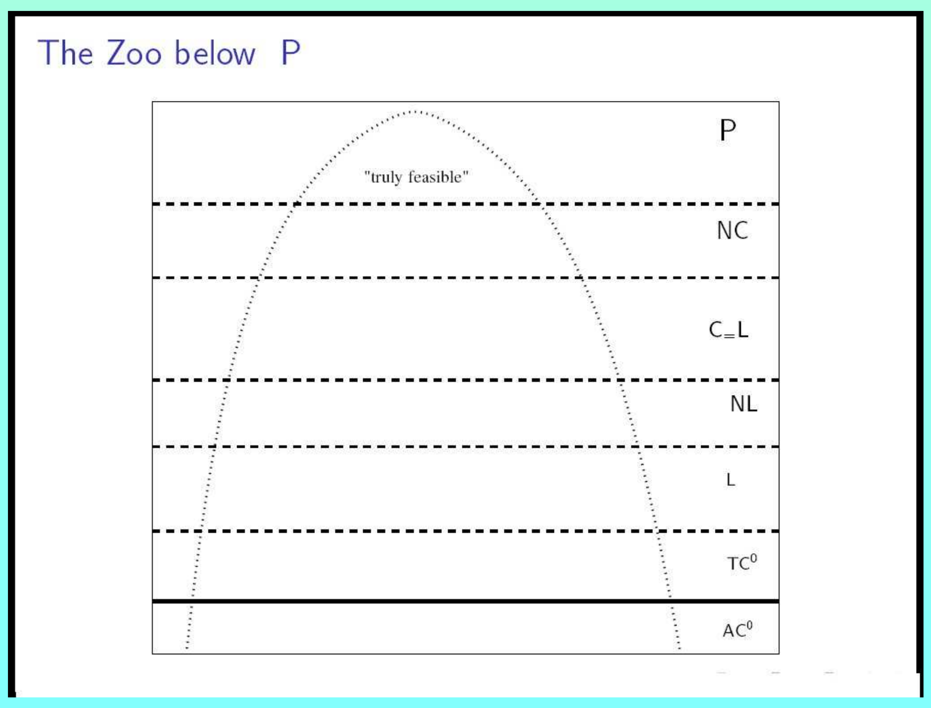
Computing the Rank

- The naive approach : EXPonential time.
- Can be solved in Polynomial time; Gaussian elimination : inherently sequential.
- Rank can be computed in NC. Elegant parallel algorithm ([Mul87]) by relating the problem to testing if some coefficients of the characterstic polynomial are zeros.
- Refined complexity bounds by [ABO99]. Upperbound testing is complete for $\mathsf{C_{=}L}$.

Complexity Theory Preliminaries

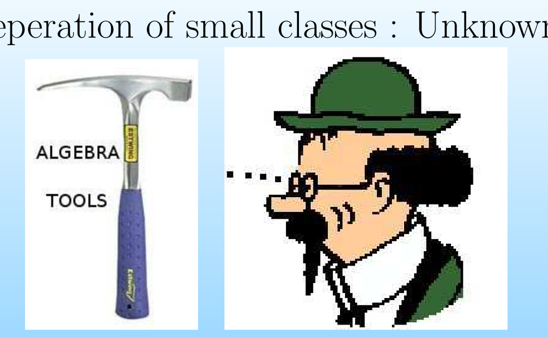
Classification of problems in P solved by various resource bounded models of computation.

Class	Resource Bound	Complete Problem
L	log space deterministic TM	Reachability in undirected forests
NL	log space nondeterminsitic TM	Reachability in directed graphs
$\mathsf{C_{=}L}$	log space "balanced" NTM	Singularity of boolean matrices
$\mathsf{AC^0}$	poly size, constant depth circuits	Reachability in const. width maze
$\mathsf{TC^1}$	$\mathsf{AC^0}$ + "majority"	Testing Majority



Turing/Circuit Model : Combinatorial !

Seperation of small classes : Unknown



Rank Computation : Algebraic !

Characterising computation might help.

Several applications do have inherent structure for the matrices.

$M = [a_{i,j}]$ is **diagonally dominant** if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$$

Fun fact : If for all i, the dominance is strict then M in non-singular.

Matrix type	RANK BOUND	SINGULAR
General	$\mathsf{C_{=}L}$-complete [ABO99]	$\mathsf{C_{=}L}$-complete [ABO99]
Sym.Non-neg.	$\mathsf{C_{=}L}$-complete [ABO99]	$\mathsf{C_{=}L}$-complete [ABO99]
Sym.Non-neg. Diag. Dom.	L-complete	L-complete
Diagonal	$\mathsf{TC^0}$-complete	in $\mathsf{AC^0}$

How close is M to a rank r matrix?
Given a matrix M and $r < n$, **rigidity** of the matrix M ($R_M(r)$) is the number of entries of the matrix that we need to change to bring the rank below r

- A natural linear algebraic optimisation problem, with important applications in control theory.
- Interesting in a circuit complexity theory setting. Highly rigid linear transformations (matrices) have some "nice" size-depth tradeoff in circuits computing them [Val77].

Computing Rigidity

RIGID(M, r, k): Given a matrix M, values r and k, is $R_M(r) \leq k$?.

- Over any finite field $\mathbb{F}$, RIGID is in NP. Over $\mathbb{F}_2$, RIGID is NP-hard too [Des] : reduction from Nearest Neighbour Decoding problem.
- Over infinite fields we don't even know if it is decidable.
- If k is constant, restriced to boolean matrices, RIGID is $\mathsf{C_{=}L}$-complete.

In many applications, the amount of change of the matrix entries dictates the cost. So we would like the changes to be small.

Bounded Rigidity
Given a matrix M and $r < n$ and b, **bounded rigidity** of the matrix M ($R_M(b, r)$) is the number of entries of the matrix that we need to change to bring the rank below r, if the change allowed per entry is atmost b.

$R_M(b, r)$ need not always exist ! Consider,

$$\begin{bmatrix} 2^k & 0 & 0 & 0 \\ 0 & 2^k & 0 & 0 \\ 0 & 0 & 2^k & 0 \\ 0 & 0 & 0 & 2^k \end{bmatrix} \quad R_M(b, n-1) \text{ doesn't exist, unless } b \geq \frac{2^k}{n}.$$

Question : Can we test this? i.e, for a given matrix M, bound b, target rank r, can we efficiently test, whether $R_M(b, r)$ exists ?
It is NP-hard for arbitrary r and NP-complete for the case of singularity.

Essential Ideas..

Equivalent formulation : Define an interval of matrices $[A]$ where $m_{ij} - b \leq a_{ij} \leq m_{ij} + b$
Question : Is there a rank r matrix $B \in [A]$ such that $M - B$ has atmost k non-zero entries?
The bound b defines an interval for each entry of the matrix. The determinant is a multlinear polynomial on the entries of the matrix. Now use the following lemma:

Zero-on-an-edge Lemma
For a multilinear polynomial p on t variables, consider the t-dimensional hypercube defined by the interval of each of the variables. If there is a zero of the polynomial in the hypercube then there is a zero on an edge of the hypercube.

NP algorithm : Guess the "nice" singular matrix and verify. Hardness: A reduction from MAX-CUT problem.

Future Work/Open Problems

- Are there characterisations of other small complexity classes (like NL) using the rank/determinant computation?
- Is there a recursive(or better) upperbound for rigidity over infinite fields?
- Can bounded rigidity be decided in NP? - is there a generalisation of the zero-on-an-edge lemma to arbitrary rank?

References

[ABO99] E. Allender, R. Beals, and M. Ogihara. The complexity of matrix rank and feasible systems of linear equations. In *Computational Complexity, 8, 99-126, 1999*, 1999.

[Mul87] K. Mulmuley. A fast parallel algorithm to compute the rank of a matrix over an arbitrary field. *Combinatorica*, 7:101–104, 1987.

[Val77] L. G. Valiant. Graph theoretic arguments in low-level complexity. In *MFCS*, 1977.

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‡ Full version available as ECCC technical report at http://eccc.hpi-web.de/eccc-reports/2006/TR06-100/