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Proofs That Really Count

The Art of Combinatorial Proof

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Foreword

Every proof in this book is ultimately reduced to a counting problem—typically enumerated in two different ways. Counting leads to beautiful, often elementary, and very concrete proofs. While not necessarily the simplest approach, it offers another method to gain understanding of mathematical truths. To a combinatorialist, this kind of proof is the *only* right one. We offer *Proofs That Really Count* as the counting equivalent of the visual approach taken by Roger Nelsen in *Proofs Without Words I & II* [37, 38].

Why count?

As human beings we learn to count from a very early age. A typical 2 year old will proudly count to 10 for the coos and applause of adoring parents. Though many adults readily claim ineptitude in mathematics, no one ever owns up to an inability to count. Counting is one of our first tools, and it is time to appreciate its full mathematical power. The physicist Ernst Mach even went so far as to say, "There is no problem in all mathematics that cannot be solved by direct counting" [36].

Combinatorial proofs can be particularly powerful. To this day, I (A.T.B.) remember my first exposure to combinatorial proof when I was a freshman in college. My professor proved the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

by writing

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$$

and asking "In how many ways can we create an $x^k y^{n-k}$ term?" Sudden clarity ensued. The theorem made perfect sense. Yes, I had seen proofs of the Binomial Theorem before, but they had seemed awkward and I wondered how anyone in his or her right mind would create such a result. But now it seemed very natural. It became a result I would never forget.

What to count?

We have selected our favorite identities using numbers that arise frequently in mathematics (binomial coefficients, Fibonacci numbers, Stirling numbers, etc.) and have chosen elegant counting proofs. In a typical identity, we pose a counting question, and then answer it in two different ways. One answer is the left side of the identity; the other answer is the right side. Since both answers solve the same counting question, they must be equal. Thus the identity can be viewed as a counting problem to be tackled from two different angles.

We use the identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

to illustrate a proof structure found throughout this book. There is no need to use the formula $\frac{n!}{k!(n-k)!}$ for $\binom{n}{k}$. Instead, we interpret $\binom{n}{k}$ as the number of k-element subsets of an n-element set, or more colorfully, as the number of ways to select a committee of k students from a class of n students.

Question: From a class of n students, how many ways can we create a committee?

Answer 1: The number of committees with 0 students is $\binom{n}{0}$. The number of committees with 1 student is $\binom{n}{1}$. In general, the number of committees with exactly k students is $\binom{n}{k}$. Hence the total number of committees is $\sum_{k=0}^{n} \binom{n}{k}$.

Answer 2: To create a committee of arbitrary size, we decide, student by student whether or not they will be on the committee. Since each of the n students is either "on" or "off" the committee, there are 2 possibilities for each student and thus 2^n ways to create a committee.

Since our logic is impeccable in both answers, they must be equal, and the identity follows.

Another useful proof technique is to interpret the left side of an identity as the size of a set, the right side of the identity as the size of a different set, and then find a one-to-one correspondence between the two sets. We illustrate this proof structure with the identity

$$\sum_{k\geq 0} \binom{n}{2k} = \sum_{k\geq 0} \binom{n}{2k+1} \quad \text{for } n > 0.$$

Both sums are finite since $\binom{n}{i} = 0$ whenever $i > n$. Here it is easy to see *what* both sides count. The challenge is to find the correspondence between them.

Set 1: The committees with an even number of members formed from a class of n students. This set has size $\sum_{k\geq 0} \binom{n}{2k}$.

Set 2: The committees with an odd number of members formed from a class of n students. This set has size $\sum_{k\geq 0} \binom{n}{2k+1}$.

Correspondence: Suppose one of the students in the class is named Waldo. Any committee with an even number of members can be turned into a committee with an odd number of members by asking "Where's Waldo?" If Waldo is on the committee, then remove him. If Waldo is not on the committee, then add him. Either way, the parity of the committee has changed from even to odd.

Since the process of "removing or adding Waldo" is completely reversible, we have a one-to-one correspondence between these sets. Thus both sets must have the same size, and the identity follows.

Often we shall prove an identity more than one way, if we think a second proof can bring new insight to the problem. For instance, the last identity can be handled by counting the number of even subsets directly. See Identity 129 and the subsequent discussion.

What can you expect when reading this book? Chapter 1 introduces a combinatorial interpretation of Fibonacci numbers as square and domino tilings, which serves as the foundation for Chapters 2–4. We begin here because Fibonacci numbers are intrinsically interesting and their interpretation as combinatorial objects will come as a delightful surprise to many readers. As with all the chapters, this one begins with elementary identities and simple arguments that help the reader to gain a familiarity with the concepts before proceeding to more complex material. Expanding on the Fibonacci tilings will enable us to explore identities involving generalized Fibonacci numbers including Lucas numbers (Chapter 2), arbitrary linear recurrences (Chapter 3), and continued fractions (Chapter 4.)

Chapter 5 approaches the traditional combinatorial subject of binomial coefficients. Counting sets with and without repetition leads to identities involving binomial coefficients. Chapter 6 looks at binomial identities with alternating signs. By finding correspondences between sets with even numbers of elements and sets with odd numbers of elements, we avoid using the familiar method of overcounting and undercounting provided by the Principle of Inclusion-Exclusion.

Harmonic numbers, like continued fractions, are not integral—so a combinatorial explanation requires investigating the numerator and denominator of a particular representation. Harmonic numbers are connected to Stirling numbers of the first kind. Chapter 7 investigates and exploits this connection in addition to identities involving Stirling numbers of the second kind.

Chapter 8 considers more classical results from arithmetic, number theory, and algebra including the sum of consecutive integers, the sum of consecutive squares, sum of consecutive cubes, Fermat's Little Theorem, Wilson's Theorem, and a partial converse to Lagrange's Theorem.

In Chapter 9, we tackle even more complex Fibonacci and binomial identities. These identities require ingenious arguments, the introduction of colored tiles, or probabilistic models. They are perhaps the most challenging in the book, but well worth your time.

Occasionally, we digress from identities to prove fun applications. Look for a divisibility proof on Fibonacci numbers in Chapter 1, a magic trick in Chapter 2, a shortcut to calculate the parity of binomial coefficients in Chapter 5 and generalizations to congruences modulo arbitrary primes in Chapter 8.

Each chapter, except the last, includes a set of exercises for the enthusiastic reader to try his or her own counting skills. Most chapters contain a list of identities for which combinatorial proofs are still being sought. Hints and references for the exercises and a complete listing of all the identities can be found in the appendices at the end of the book.

Our hope is that each chapter can stand independently, so that you can read in a nonlinear fashion if desired.

Who should count?

The short answer to this question is "Everybody counts!" We hope this book can be enjoyed by readers without special training in mathematics. Most of the proofs in this book can be appreciated by students at the high school level. On the other hand, teachers may find this book to be a valuable resource for classes that emphasize proof writing and creative problem solving techniques. We do not consider this book to be a complete survey of combinatorial proofs. Rather, it is a beginning. After reading it, you will never view quantities like Fibonacci numbers and continued fractions the same way again. Our hope is that an identity like

Identity 5. $$f_{2n+1} = \sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n-i}{j}\binom{n-j}{i}$$

for Fibonacci numbers should give you the feeling that something is being counted and the desire to count it. Finally, we hope this book will serve as an inspiration for mathematicians who wish to discover combinatorial explanations for old identities or discover new ones. We invite you, our readers, to share your favorite combinatorial proofs with us for (possible) future editions.

After all, we hope all of our efforts in writing this book will count for something.

Who counts?

We are pleased to acknowledge the many people who made this book possible—either directly or indirectly.

Those who came before us are responsible for the rise in popularity of combinatorial proof. Books whose importance cannot be overlooked are *Constructive Combinatorics* by Dennis Stanton and Dennis White, *Enumerative Combinatorics Volumes 1 & 2* by Richard Stanley, *Combinatorial Enumeration* by Ian Goulden and David Jackson, and *Concrete Mathematics* by Ron Graham, Don Knuth & Oren Patashnik. In addition to these mathematicians, others whose works continue to inspire us include George E. Andrews, David Bressoud, Richard Brualdi, Leonard Carlitz, Ira Gessel, Adriano Garsia, Ralph Grimaldi, Richard Guy, Stephen Milne, Jim Propp, Marta Sved, Herbert Wilf, and Doron Zeilberger.

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CHAPTER 5

Binomial Identities

Definition The *binomial coefficient* $\binom{n}{k}$ is the number of k-element subsets of $\{1,\ldots,n\}$.

Definition The *multichoose coefficient* $\left(\!\binom{n}{k}\!\right)$ is the number of k-element multisubsets of $\{1,\ldots,n\}$.

Examples of binomial coefficients are $\binom{4}{0}=1$, $\binom{4}{1}=4$, $\binom{4}{2}=6$, $\binom{4}{3}=4$, and $\binom{4}{4}=1$. Examples of multichoose coefficients are $\left(\!\binom{4}{0}\!\right)=1$, $\left(\!\binom{4}{1}\!\right)=4$, $\left(\!\binom{4}{2}\!\right)=10$, $\left(\!\binom{4}{3}\!\right)=20$, and $\left(\!\binom{4}{4}\!\right)=35$.

5.1 Combinatorial Interpretations of Binomial Coefficients

Binomial coefficients were born to count! Unlike most of the quantities we have discussed in this book, binomial coefficients are almost always defined as the answer to a counting problem. Specifically, we define $\binom{n}{k}$ to be the number of k-element subsets of $\{1,2,\ldots,n\}$. Put another way, $\binom{n}{k}$ counts the ways to select a committee of k students from a class of n students where the order of the selection is not important. By definition we have, for $n \geq 0$, $\binom{n}{0}=1$, and for $k<0$, $\binom{n}{k}=0$. (Although it's possible to define $\binom{n}{k}$ for negative values of n, we will not do so here.)

Binomial coefficients have a simple algebraic formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{5.1}$$

which can be easily seen by the following identity:

Identity 125 *For* $0 \leq k \leq n$, $n! = \binom{n}{k}k!(n-k)!$

Question: How many ways can the numbers 1 through n be arranged in a list?

Answer 1: There are $n!$ arrangements since the first number can be chosen n ways, the next number can be chosen $n-1$ ways, and so on. (We shall have more to say about $n!$ in Chapter 7.)

Answer 2: Condition on which numbers are among the first k in our arrangement. There are, by definition, $\binom{n}{k}$ ways to choose which of the n numbers appear among the first k. Once these are chosen, there are $k!$ ways to arrange them, followed by $(n-k)!$ ways to arrange the remaining elements. Hence the numbers 1 through n can be arranged in $\binom{n}{k}k!(n-k)!$ ways.

We shall take pains to avoid invoking equation (5.1), in the same way that we avoided using Binet's formula (Identity 240) when proving identities for Fibonacci numbers in Chapter 1. Our goal is to understand binomial identities entirely from their combinatorial definition and to avoid algebraic arguments (such as proofs by induction) as much as possible.

5.2 Elementary Identities

In this section, we present simple combinatorial proofs of binomial coefficient identities. Although the arguments we present in this section are quite well-known, they are beautiful nonetheless. In subsequent sections of this chapter, the arguments will become trickier.

Identity 126 *For* $0 \le k \le n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Question: How many ways can we create a size k committee of students from a class of n students?

Answer 1: By definition, $\binom{n}{k}$.

Answer 2: We may choose $n-k$ students to exclude from the committee, which can be done $\binom{n}{n-k}$ ways.

Identity 127 *For* $0 \le k \le n$, *(except* $n = k = 0$*)*,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Question: How many ways can we create a size k committee of students from a class of n students?

Answer 1: As before, $\binom{n}{k}$.

Answer 2: Condition on whether or not student n is on the committee. There are $\binom{n-1}{k}$ committees that exclude student n, and $\binom{n-1}{k-1}$ committees that include student n.

Identity 127 (along with initial conditions $\binom{0}{0} = 1$ and $\binom{n}{k} = 0$ for $n < k$) can be used to generate binomial coefficients in a convenient table known as Pascal's Triangle. See Figure 5.1.

Although the previous identities are easy to prove by using the algebraic formula for $\binom{n}{k}$ given in Identity 125, the next identity is not at all obvious from the factorial definition of $\binom{n}{k}$. Note that the sum on the left is finite, since $\binom{n}{k} = 0$ for $k > n$.

Identity 128 *For* $n \ge 0$,

$$\sum_{k \ge 0} \binom{n}{k} = 2^n.$$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Figure 5.1. Numbering our rows and columns with nonnegative integers, the number in row n and column k is $\binom{n}{k}$, and all missing entries are zero.

Question: How many ways can we create a committee (of any size) from a class of n students?

Answer 1: Since for $0 \le k \le n$, there are $\binom{n}{k}$ committees of size k, there are $\sum_{k\ge 0}\binom{n}{k}$ such committees.

Answer 2: Decide, student by student, whether or not to put that student on the committee. Since there are two possibilities for each student (on or off), there are 2^n possible committees.

Identity 129 *For* $n \ge 1$,

$$\sum_{k\ge 0}\binom{n}{2k} = 2^{n-1}.$$

Question: How many ways can we create a committee with an even number of members from a class of n students?

Answer 1: Since for $0 \le 2k \le n$, there are $\binom{n}{2k}$ committees of size $2k$, there are $\sum_{k\ge 0}\binom{n}{2k}$ such committees.

Answer 2: The first $n-1$ students can be freely chosen to be on or off of the committee, as in the previous proof. Once these choices are made, then the fate of the nth student is completely determined so that the final committee size is an even number. Consequently, there are 2^{n-1} such committees.

Notice that the last two identities imply that exactly half of all subsets of $\{1, \ldots, n\}$ are even. Consequently, half of them must also be odd. Equivalently, this says

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0.$$

We shall have more to say about such alternating sums in the next chapter.

Identity 130 *For* $0 \le k \le n$,

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Question: How many ways can we create a size k committee of students from a class of n students, where one of the committee members is designated as chair?

Answer 1: There are $\binom{n}{k}$ ways to choose the committee, then k ways to select the chair. Hence there are $k\binom{n}{k}$ possible outcomes.

Answer 2: First select the chair from the class of n students. Then from the remaining $n-1$ students, pick the remaining $k-1$ committee members. This can be done $n\binom{n-1}{k-1}$ ways.

The next identity can be treated as a continuation of Identity 130.

Identity 131 *For* $n \geq 1$,

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Question: How many ways can we create a committee (of any size) from a class of n students, where one of the committee members is designated as chair?

Answer 1: For a committee of size k, where $0 \leq k \leq n$, there are $k\binom{n}{k}$ such committees. Altogether, we have $\sum_{k=0}^{n} k\binom{n}{k}$ possible outcomes.

Answer 2: First select the chair from the class of n students. Then from the remaining $n-1$ students, there are 2^{n-1} ways to choose a subset of them to form the rest of the committee.

Dividing both sides of the last identity by 2^n allows us to give a different combinatorial proof of the equivalent identity:

$$\frac{\sum_{k=0}^{n} k\binom{n}{k}}{2^n} = \frac{n}{2}.$$

Question: What is the average size of a subset of $\{1, 2, \ldots, n\}$?

Answer 1: We add up the sizes of all subsets and divide by the total number of subsets. Since for $0 \leq k \leq n$, there are $\binom{n}{k}$ subsets of size k, and there are 2^n subsets altogether, the average subset size is $\frac{\sum_{k=0}^{n} k\binom{n}{k}}{2^n}$.

Answer 2: Pair up each subset with its complement. Since each such pair has n elements, each complementary pair has an average of $\frac{n}{2}$ elements. Hence the average subset size is $\frac{n}{2}$.

The next identity, *Vandermonde's Identity*, has a simple combinatorial interpretation.

Identity 132 *For* $m \geq 0$, $n \geq 0$,

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}.$$

Question: From a class of $m+n$ students, consisting of m men and n women, how many ways can one form a size k committee?

Answer 1: By definition, $\binom{m+n}{k}$.

Answer 2: Condition on the number of men on the committee. For $0 \leq j \leq k$, we can form a committee with j men by first choosing the men ($\binom{m}{j}$ ways), then the remaining $k-j$ committee members can be chosen from the women in $\binom{n}{k-j}$ ways. Altogether, there are $\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$ such committees.

Many of the previous identities can be proved using algebraic methods based on the *Binomial Theorem*, but even that can be proved combinatorially.

Identity 133 *For* $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Question: In a class of n students, each student is given the choice of solving either one of x different algebra problems or one of y different geometry problems. How many different outcomes are possible?

Answer 1: Since each student has $x+y$ choices for which problem to solve, there are $(x+y)^n$ possible outcomes.

Answer 2: Condition on the number of students who choose to solve an algebra problem. For $0 \leq k \leq n$, there are $\binom{n}{k}$ ways to determine which k students chose to do an algebra problem, then x^k ways for them to decide which algebra problems to do, then y^{n-k} ways for the remaining $n-k$ students to decide which geometry problems to do. Altogether, there are $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ possible outcomes.

The proof above assumes that x and y are integers, although the theorem is true for real or complex values of x and y as well. There are several combinatorial ways around this issue. One way is to observe for any fixed y, both sides of the identity are degree n polynomials in x that agree on an infinite number of points. Hence they must be equal.

Another (slightly more algebraic) way to view this identity is to think of the expression

$$(x+y)^n = (x+y)(x+y)\cdots(x+y) \qquad (n \text{ times}),$$

and ask, "How many ways can one create an $x^k y^{n-k}$ term?" Each such term arises by choosing an x term from k of the $x+y$ factors, which can be done $\binom{n}{k}$ ways.

The next identity has an interesting application to number theory.

Identity 134 *For* $0 \leq m \leq k \leq n$,

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Question: In a class of n students, how many ways can we choose a size k committee that contains a size m subcommittee?

Answer 1: The committee can be chosen $\binom{n}{k}$ ways, then the subcommittee can be chosen $\binom{k}{m}$ ways.

Answer 2: First choose the m students who will be on the committee and the subcommittee. This can be done $\binom{n}{m}$ ways. From the remaining $n-m$ students, the $k-m$ students to be on the committee but not the subcommittee can be chosen $\binom{n-m}{k-m}$ ways.

As a simple consequence of this last identity, Erdős and Szekeres proved the following simple fact about binomial coefficients. (It seems that this was not known prior to 1978!)

Corollary 7 *For $0 < m \le k < n$, $\binom{n}{m}$ and $\binom{n}{k}$ have a nontrivial common factor. That is,* $\gcd(\binom{n}{m}, \binom{n}{k}) > 1$.

Proof. Suppose, to the contrary, that $\binom{n}{m}$ and $\binom{n}{k}$ are relatively prime. By Identity 134, $\binom{n}{m}$ divides $\binom{n}{k}\binom{k}{m}$. But since $\binom{n}{m}$ and $\binom{n}{k}$ have no common factors, it follows that $\binom{n}{m}$ divides $\binom{k}{m}$. This is impossible, since it is (combinatorially) clear that $\binom{n}{m}$ is greater than $\binom{k}{m}$. ◇

5.3 More Binomial Coefficient Identities

For the identities in this section, it is more convenient to talk about subsets than committees. While Identity 128 proved that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$, no general closed form exists for the partial sum $\sum_{k=0}^{m}\binom{n}{k}$ where $m < n$. However, if we interchange the roles of the fixed and the indexed variable in the binomial summation, a closed form for the partial sum does exist. Specifically:

Identity 135 *For $0 \le k \le n$,*

$$\sum_{m=k}^{n}\binom{m}{k} = \binom{n+1}{k+1}.$$

Question: How many $(k+1)$-subsets are contained in the set $\{1, 2, \ldots, n+1\}$?

Answer 1: By definition, $\binom{n+1}{k+1}$.

Answer 2: Condition on the largest number in the subset. A size $k+1$ subset with maximum element $m+1$ can be created $\binom{m}{k}$ ways. Since $m+1$ can be as small as $k+1$ and as large as $n+1$, there are $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k}$ subsets in total.

Identity 136 *For $0 \le k \le n/2$,*

$$\sum_{m=k}^{n-k}\binom{m}{k}\binom{n-m}{k} = \binom{n+1}{2k+1}.$$

Question: How many $(2k+1)$-subsets are contained in the set $\{1, \ldots, n+1\}$?

Answer 1: By definition, $\binom{n+1}{2k+1}$.

Answer 2: Condition on the median number in the subset. In a size $2k+1$ subset, the median element will be the $(k+1)$st smallest element, with k elements below it and k elements above it. (For example, in the set $\{2, 3, 5, 8, 13\}$, the median element is 5.) Hence, the number of size $2k+1$ subsets with median element $m+1$ is $\binom{m}{k}\binom{n-m}{k}$. Since $m+1$ can range from $k+1$ to $n+1-k$, the identity follows.

By conditioning on the rth element of the set, we obtain the following generalization.

Identity 137 *For* $1 \le r \le k$,

$$\sum_{j=r}^{n+r-k} \binom{j-1}{r-1}\binom{n-j}{k-r} = \binom{n}{k}.$$

As we have seen before, binomial coefficients and Fibonacci numbers can't help running into each other. The next few identities are variations on the same theme.

Identity 138 *For* $t \ge 1, n \ge 0$,

$$\sum_{x_1\ge 0}\sum_{x_2\ge 0}\cdots\sum_{x_t\ge 0}\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t} = f_{t+1}^n.$$

Question: In how many ways can you create subsets $S_1, S_2, \ldots, S_t$, where $S_1 \subseteq \{1,2,\ldots,n\}$ and for $2 \le i \le t$, $S_i \subseteq \{1,2,\ldots,n\}$ and S_i is disjoint from S_{i-1}?

Answer 1: Condition on the size of each subset S_i. To create subsets that are "consecutively disjoint" with sizes $x_i = |S_i|$, $1 \le i \le n$, there are $\binom{n}{x_1}$ ways to create S_1. Then, since S_2 is disjoint from S_1, there are $\binom{n-x_1}{x_2}$ ways to create S_2. Since S_3 is disjoint from S_2, there are $\binom{n-x_2}{x_3}$ ways to create S_3 and so on. Thus there are $\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t}$ ways to create $S_1,\ldots,S_t$ with respective sizes $x_1,\ldots,x_t$. Altogether $S_1, S_2, \ldots, S_t$ can be created in

$$\sum_{x_1\ge 0}\sum_{x_2\ge 0}\cdots\sum_{x_t\ge 0}\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t}$$

ways.

Answer 2: For each element $j \in \{1,\ldots,n\}$, decide which subsets contain j. By construction, the subsets containing j must be nonconsecutive. Exercise 1 in Chapter 1 shows that there are f_{t+1} ways to select the nonconsecutive subsets containing j, among the sets $S_1,\ldots,S_t$. Hence the elements 1 through n can be placed into subsets in f_{t+1}^n ways.

For those that prefer the tiling approach from Chapter 1, here is a different proof of Identity 138.

Question: In how many ways can you create n square-domino tilings $T_1,\ldots,T_n$, each of length $t+1$?

Answer 1: Each tiling can be created f_{t+1} ways, so there are f_{t+1}^n such tilings.

Answer 2: For each cell j, $1 \le j \le t$, let x_j denote the number of tilings that have a domino beginning at cell j. Conditioning on all possible values of $x_1,\ldots,x_t$, we have $\binom{n}{x_1}$ ways to decide which of $T_1,\ldots,T_n$ begin with a domino. (The rest begin with a square.) Among the $n-x_1$ tilings that do not begin with a domino, there are $\binom{n-x_1}{x_2}$ ways to choose which tilings have a domino beginning at cell 2. (Among these $n-x_1$ tilings, the unchosen ones have a square at cell 2.) Among the $n-x_2$ tilings that do not have a domino covering cells 2 and 3, there are $\binom{n-x_2}{x_3}$ ways to choose which tilings have a domino beginning at cell 3. Continuing in this fashion, $T_1,\ldots,T_n$ can be tiled in $\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t}$ ways, as desired.

Generalizing the previous argument, we obtain

Identity 139 *For $t \geq 1, n \geq 0, c \geq 0$,*

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n-c}{x_1} \binom{n-x_1}{x_2} \binom{n-x_2}{x_3} \cdots \binom{n-x_{t-1}}{x_t} = f_t^c f_{t+1}^{n-c}.$$

Question: In how many ways can you create n square-domino tilings $T_1, \ldots, T_n$ of length $t+1$, where $T_1, \ldots, T_c$ must begin with a square?

Answer 1: There are $f_t^c f_{t+1}^{n-c}$ such tilings, since the first c t-tilings can be created f_t ways, and the remaining $n-c$ $(t+1)$-tilings can be created f_{t+1}^{n-c} ways.

Answer 2: The exact same reasoning as in the last proof applies here. The only difference is that the x_1 tilings that begin with dominoes must be chosen from $T_{c+1}, \ldots, T_n$. Hence the first step can be performed $\binom{n-c}{x_1}$ ways instead of $\binom{n}{x_1}$.

Identity 138 can be generalized in a different direction to produce a Lucas identity.

Identity 140 *For $t \geq 1, n \geq 0$,*

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-x_2}{x_3} \cdots \binom{n-x_{t-1}}{x_t} 2^{x_1} = L_{t+1}^n.$$

The proof is the same as in Identity 138, but now each of the x_1 tilings that begin with a domino is given one of two phases. Even more generally, we have

Identity 141 *For $t \geq 1, n \geq 0$,*

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-x_2}{x_3} \cdots \binom{n-x_{t-1}}{x_t} G_0^{x_1} G_1^{n-x_1} = G_{t+1}^n,$$

where G_j is the jth element of the Gibonacci sequence beginning with G_0 and G_1.

We remark that Identity 138 and its generalizations arose from our attempts to combinatorially prove the following generalization of Identity 5 from Chapter 1.

Identity 142 *For $t \geq 1, n \geq 0$,*

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n-x_t}{x_1} \binom{n-x_1}{x_2} \binom{n-x_2}{x_3} \cdots \binom{n-x_{t-1}}{x_t} = \frac{f_{tn+t-1}}{f_{t-1}}.$$

For a combinatorial proof of that, see [14].

5.4 Multichoosing

In this section, we examine identities involving the quantity $\left(\!\binom{n}{k}\!\right)$, spoken "n multichoose k", which counts the ways to select k objects from a set of n elements, where order is not important, but repetition is allowed. The 20 possible *multisubsets* of size 3 that can be created from $\{1, 2, 3, 4\}$ are illustrated in Figure 5.2. By contrast $\binom{n}{k}$ counts the same

5.7 Exercises

Prove each of the identities below by a direct combinatorial argument.

Identity 151 *For* $n \geq k \geq 0$, $(n-k)\binom{n}{k} = n\binom{n-1}{k}$.

Identity 152 *For* $n \geq 2$, $k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$.

Identity 153 *For* $n \geq 3$, $\sum_{k\geq 0} k(k-1)(k-2)\binom{n}{k} = n(n-1)(n-2)\binom{n-3}{3}$.

Identity 154 *For* $n \geq 4$, $\binom{\binom{n}{2}}{2} = 3\binom{n}{4} + 3\binom{n}{3}$.

Identity 155 *For* $0 \leq m \leq n$, $\sum_{k\geq 0}\binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}$.

Identity 156 *For* $0 \leq m < n$, $\sum_{k\geq 0}\binom{n}{2k}\binom{2k}{m} = \binom{n}{m}2^{n-m-1}$.

Identity 157 *For* $m, n \geq 0$, $\sum_{k\geq 0}\binom{m}{k}\binom{n}{k} = \binom{m+n}{n}$.

Identity 158 *For* $m, n \geq 0$, $\sum_{k\geq 0}\binom{n}{k}\binom{n-k}{m-k} = \binom{n}{m}2^{m}$.

Identity 159 *For* $n \geq 1$, $\sum_{k\geq 0} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.

Identity 160 *For* $n \geq 0$, $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$.

Identity 161 *For* $n \geq 0$, $\sum_{k\geq 0}\binom{n}{2k}\binom{2k}{k}2^{n-2k} = \binom{2n}{n}$.

The next identity can be proved using binomial or multinomial coefficient interpretations.

Identity 162 *For* $m, n \geq 0$, $\sum_{k=0}^{m}\binom{n+k}{k} = \binom{n+m+1}{m}$.

Identity 163 *For* $t \geq 1, 0 \leq c \leq n$, $(G_1 f_t)^c G_{t+1}^{n-c}$ *equals*

$$\sum_{x_1\geq 0}\sum_{x_2\geq 0}\cdots\sum_{x_t\geq 0}\binom{n-c}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t}G_0^{x_1}G_1^{n-x_1}$$

where G_j *is the* j*th element of the Gibonacci sequence beginning with* G_0 *and* G_1.

Identity 164 *For* $n, k \geq 0$, $\left(\!\binom{n}{2k+1}\!\right) = \sum_{m=1}^{n}\left(\!\binom{m}{k}\!\right)\left(\!\binom{n-m+1}{k}\!\right)$.

Identity 165 *For* $n \geq 0$, $\sum_{k=0}^{n}\binom{n+k}{2k} = f_{2n}$.

Identity 166 *For* $n \geq 1$, $\sum_{k=0}^{n-1}\binom{n+k}{2k+1} = f_{2n-1}$.

Other Exercises

1. Prove for $n \geq 0, m \geq 1$, that $\sum_{k\geq 0} k\binom{n}{k}\binom{m}{m-k} = n\binom{n+m-1}{m-1}$. Then apply the same logic to arrive at a closed form for $\sum_{k\geq 0}\binom{k}{r}\binom{n}{k}\binom{m}{m-k}$.

2. Many combinatorial proofs for binomial coefficients can also be done by *path counting*. Prove that the number of ways to walk from the point $(0,0)$ to the point (a,b) such that every step is one unit to the right or one unit up is $\binom{a+b}{a}$.

3. Combinatorially prove the identities below by path counting arguments.

   (a) For $a, b > 0$, $\binom{a+b}{a} = \binom{a+b-1}{a} + \binom{a+b-1}{a-1}$.

   (b) For $a, b \geq 0$, $\sum_{k=0}^{a}\binom{a}{k}\binom{b}{a-k} = \binom{a+b}{a}$.

   (c) For $0 \leq s \leq a$, $\sum_{k=0}^{s}\binom{s}{k}\binom{a+b-s}{a-k} = \binom{a+b}{a}$.

   (d) For $a, b \geq 0$, $\sum_{k=0}^{b}\binom{a+k}{a} = \binom{a+b+1}{a+1}$.

   (e) For $0 \leq s \leq a$ and $b \geq 0$, $\sum_{m=s}^{s+b}\binom{m}{s}\binom{a+b-m}{a-s} = \binom{a+b+1}{a}$.

   (f) This last identity only looks simple. For $n \geq 0$, $\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$.

4. Catalan numbers. Prove that the number of paths from $(0,0)$ to $(2n,n)$ that never go above the main diagonal $y = x$ is $\frac{1}{n+1}\binom{2n}{n}$.

5. Partitions of integers. Let $\pi(n)$ count the ways that the integer n can be expressed as the sum of positive integers, written in non-increasing order. Thus $\pi(4) = 5$, since 4 can be expressed as $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$. Prove that the number of integer partitions with at most a positive parts, all of which are at most b, is $\binom{a+b}{a}$. (Example: When $a = 2, b = 3$, the ten partitions are: $3+3, 3+2, 3+1, 3, 2+2, 2+1, 2, 1+1, 1, \phi$.)

6. An ordered partition or (composition) of n does not require the summands to be in non-increasing order. For instance 4 has eight ordered partitions: $4 = 3+1 = 1+3 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1$. Prove that the number of ordered partitions of n with exactly k parts is $\binom{n-1}{k-1}$ and the total number of ordered partitions of n is 2^{n-1}.