Unique Games Conjecture and Hardness of Approximation

Anup Joshi

Indian Institute of Technology, Madras

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Recap

Theorem (The PCP Theorem)

$\mathrm{NP} = \mathrm{PCP}(O(\log n), O(1))$

$$L \in \mathrm{PCP}(O(\log n), O(1)) \Leftrightarrow L \leq GAP - qCSP$$

The PCP Theorem

Definition (PCP (Alternatively we can write))

A language K is in $PCP_{c,s}[r(n), q(n)]$ if there exists a (r(n), q(n))-restricted verifier V such that given a string $x \in \{0,1\}^n$ it satisfies,

- *Completeness* : If $x \in L$, then there is a proof $y : Pr[V^y(x) = 1] \geq c$;
- *Soundness* : If $x \notin L$, then for all $y : Pr[V^y(x) = 1] < s$

where the probabilities are taken over V's choice of random bits and $0 \leq s < c \leq 1$. Also, $|y| \leq q(n).2^{r(n)}$.

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- Important for relating PCP and the hardness of approximation

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- Gap between the best algorithms today, and the inapproximability results. How do we get tight bounds of inapproximability?

So, Why do we care about tight hardness bounds?

2 Prover 1-round Game

Definition (2 Prover 1-round Game)

A language L is in $2P1R_{c,s}[r(n)]$ if there exists a probabilistic poly-time verifier V that uses r(n) random bits such that given a string $x \in \{0,1\}^n$ it produces two queries $q1$ and $q2$ and two provers P_1 and P_2 have answers $P_1(q_1)$ and $P_2(q_2)$ to queries q_1 and q_2 satisfies,

- *Completeness* : If $x \in L$, V accepts with probability c;
- *Soundness* : If $x \notin L$, for any answer of the provers, V accepts with probability at most s

for $0 \leq s < c \leq 1$

Conjecture (Unique Games Conjecture)

For arbitrarily small constants ζ, $\delta > 0$, there exists a constant $k = k(\zeta, \delta)$ such that it is NP-hard to determine whether a 2P1R game with answers from a domain of size k has value at least $1 - \zeta$ or at most δ.

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- If it is True:
  - Vertex Covering will be hard to approximate within $2 - \epsilon$
  - .879 Algorithm is the best for Max-Cut
- If it is false?

Thank You