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Lecture 18

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In the first half of this lecture, we will present a randomized log space algorithm for the connectivity problem on undirected graphs. In the second half, we will introduce the concept of expander graphs, and illustrate its applications through some examples.

1 USTCON is in RL

First, let's state the undirected u-t connectivity problem, or USTCON for short. In this problem, we are given an undirected graph $G = (V, E)$ as represented by its adjacency matrix, and two vertices u and v of the graph, which are given as two indices into the vertex set V of G. Let $n = |V|$.

To begin with, it is known that the USTCON problem is known to characterise a complexity class called Symmetric Logspace (SL). We will skip the details of this, as we will later prove a stronger result that SL = L, namely that we will give a log-space algorithm to test undirected connectivity.

But now, we will present a very simple randomized algorithm for USTCON which is basically just a plain random walk, and analyse its complexity.

1. start a random walk from s
2. if the current vertex is t, then stop and accept; otherwise, choose a random neighbor of the current vertex and set the curret vertex to that neighbor
3. repeat the previous step k times
4. after that, stop and reject

We will prove in a minute that

Proposition 1 *There is a positive constant c such that if $k \geq cn^4 \log n$, and s and t are actually connected, then*

$$Pr[\textit{the above algorithm rejects}] < \frac{1}{2}$$

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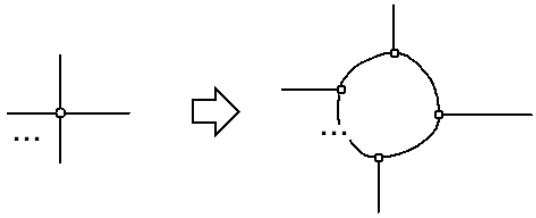

And it's trivial to show that if s and t are not connected, then the above algorithm will always reject, which means that

Theorem 2 $\mathsf{USTCON} \in \mathsf{RL}$

Now all we need to do is to justify proposition 1. In order to do that, we need to introduce some results about random walks on graphs.

To make things simpler, we assume, without loss of generality, that the graph G we are talking about is 3-regular. Actually, by performing the transformation illustrated below on every vertex of the graph, we can transform every graph into a 3-regular graph without affecting the connectivity of any pair of vertices in this graph. And it's easy to see that the size of the graph won't blow up too much because of this transformation.

Suppose A is the *normalized* adjacency matrix of G (because we are talking about a 3-regular graph, A is different from the adjacency matrix of G we commonly talk about only by a constant factor, namely, $1/3$), then starting from an initial probability distribution on the vertex set V, represented by a n-dimensional vector w, after k steps, the final distribution is $A^k w$.

An important fact about random walks on regular graphs is that the uniform distribution u is stationary. This means that $Au = u$, which is trivial to verify. And more importantly, any random walk on a non-bipartite[1], connected, regular graph will converge towards this uniform distribution. And we can define the term *mixing time* informally to mean the upper limit of the time it takes for a random walk from an arbitrary initial distribution to get "close enough" to this uniform distribution.

We can prove that for the normalized adjacency matrix of a non-bipartite, connected graph, except for a simple eigenvalue of 1 with eigenvector u, all its other eigenvalues will have absolute values less than 1. And we define the *spectral gap* λ to be the difference between 1 and the maximum absolute value of the eigenvalues other than 1. This *spectral gap* of a graph is closely related to the *mixing time* of the random walks on the graph. Actually, we can prove the following proposition

[1]normally we don't care about this because every graph can be made non-bipartite by adding self-loops on every vertex

Proposition 3 *For any positive integer k and any distribution vector w, $\|A^k w - u\|_2 \leq (1-\lambda)^k$*

Proof It's easy to show that for any distribution vector w, $w \cdot u = \frac{1}{n}$, thus $(w-u)\cdot u = 0$, that is $w - u \perp u$.

Since A has a simple eigenvalue 1, with corresponding eigenvector u, and λ is the spectral gap. By definition, we have

$$\|Aw - u\|_2 = \|A(w-u)\|_2 \leq (1-\lambda)\|w-u\|_2$$

Apply this k times, we get

$$\|A^k w - u\|_2 = \|A^k(w-u)\|_2 \leq (1-\lambda)^k \|w-u\|_2$$

and

$$\|w-u\|_2^2 = (w-u)^2 = w^2 - 2wu + u^2 = w^2 - \frac{1}{n} \leq 1$$

thus

$$\|A^k w - u\|_2 \leq (1-\lambda)^k \|w-u\|_2 \leq (1-\lambda)^k$$

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We will prove in the next lecture that

Claim 4 *For a d-regular, n-vertex graph G, its spectral gap is bounded from below, that is $\lambda \geq \frac{1}{8dn^3}$.*

This means that if we set k to be $16dn^3 \log n$, we will get

$$\begin{aligned}
\|A^k w - u\|_2 &\leq (1-\lambda)^k \\
&\leq (1 - \frac{1}{8dn^3})^{16dn^3 \log n} \\
&= \left[(1 - \frac{1}{8dn^3})^{8dn^3}\right]^{2\log n} \\
&\leq \left(\frac{1}{e}\right)^{2\log n} \\
&= n^{-2}
\end{aligned}$$

Thus we can start a random walk from an arbitrary initial distribution, and after some $O(n^3 \log n)$ steps, the chance that we hit our desired destination t (assuming that t is reachable) is at least $n^{-1} - n^{-2}$. If we repeat that $O(n)$ times, or, equivalently, just walk $O(n^4 \log n)$ steps, we will have a constant probability (say, at least $\frac{1}{2}$) of hitting t. This completes the proof for proposition 1.

2 Towards a Deterministic Logspace Algorithm for USTCON

In this section we will introduce expander graphs and Reingold's idea about showing that $\mathsf{USTCON} \in \mathsf{L}$ with the help of those special graphs.

2.1 Expander Graphs

We have two essentially equivalent ways to define expander graphs. They are called spectral expansion and combinatorial expansion

Definition 5 *A graph G is called a (n, d, λ)-spectral expander if it has n vertices, is d-regular, and has a spectral gap λ.*

Definition 6 *A graph $G = (V, E)$ is called a (n, d, α)-combinatorial expander if it has n vertices, is d-regular, and for any subset of the vertices $S \subseteq V$, if $|S| \leq \frac{1}{2}|V|$, then $|N(S)| = |\{u \in V \setminus S \ : \ u \text{ has at least one neighbor in } S\}| \geq \alpha|S|$*

We will prove the equivalence of these two definitions in the coming lectures. And we can generalize the second definition above by replacing $\frac{1}{2}$ by an arbitrary positive constant $\beta \leq \frac{1}{2}$.

It's easy to see that

Claim 7 *The diameter of a combinatorial expander graph G is in $O(\log n)$.*

2.2 Reingold's Idea about $\mathsf{USTCON} \in \mathsf{L}$

In 2004, Reingold proved that USTCON can actually be solved in L. The intuitive idea is based on the following fact

Claim 8 *The* Reachability *problem can be solved in* L *for constant-degree expander graphs.*

With this fact in mind, all we need to do now is to come up with a transformation scheme that can take an arbitrary graph and turn it into an appropriate expander graph, without blowing its size up too much, and preserve its connectivity properties. This is exactly what Reingold did in his 2004 paper. We will talk about the details of the proof in the next lecture.