Advanced Counting Techniques

CS1200, CSE IIT Madras

Meghana Nasre

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Advanced Counting Techniques

- Principle of Inclusion-Exclusion ✓
- Recurrences and its applications ✓
- Solving Recurrences

Repeated Substitution Method : Learnings

- An elementary method to solve recurrences.
  elementary does not mean simple, but a something that does not need background
- Need to observe a pattern. Do not oversimplify.
- Creativity and experience with summation of series help.
- However, the pattern has to be observed for each recurrence and there is no generic rule. Are there some recurrences that can be solved by a formula?

Ex: Solve by repeated substitution

$$T(n) = \begin{cases} 2 & \text{if } n = 0 \\ 2\sqrt{T(n-1)} & \textit{otherwise} \end{cases}$$

Sol: $T(n) = 2^{2-\frac{1}{2^n}}$

$$T(n) = \begin{cases} 12 & \text{if } n = 0 \\ 20 & \text{if } n = 1 \\ 2T(n-1) - T(n-2) & \textit{otherwise} \end{cases}$$

Sol: $T(n) = 8n + 12$

Linear recurrences

Linear Homogeneous Recurrences with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

$1 \leq i \leq k$, c_i is a real number and $c_k \neq 0$

- Linear because a_{n-1}, a_{n-2} $\ldots$ appear in separate terms and to the first power.
- Homogeneous because degree of every term is the same. There is no constant term.
- Constant coefficients because $c_1, c_2 \ldots$ are reals which do not depend on n.

Examples:

- $T(n) = 2T(n-1)$ and $T(0) = 1$.
- $T(n) = T(n-1) + T(n-2)$ and $T(0) = 0, T(1) = 1$.

Non Examples:

- $T(n) = nT(n-1)$ and $T(0) = 1$ does not have constant coefficients
- $T(n) = T(n-1) \cdot T(n-2)$ and $T(0) = 0, T(1) = 1$. not linear

Example 1

$$a_n = a_{n-1} + 2a_{n-2}$$

if base cases were $a_0 = 1, a_1 = 8$

- Is this a well defined recurrence? No! base cases are missing.
- $1, 2, 2^2, 2^3, \ldots$ is a possible solution; closed form 2^n.
- $1, -1, 1, -1, \ldots$ is another possible solution; closed form $(-1)^n$.
- None of the above are solutions.

Qn: Is it possible to make use of "some" solutions (not necessarily satisfying base cases) to get a valid solution?

Ans: Yes it is possible.

Linear Homogeneous Recurrences of degree two with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

- Degree two (or second order) says that a_n depends on two previous terms.
- We will deal with degree two recurrences initially.

Claim: If $r_0, r_1, r_2, \ldots,$ and $s_0, s_1, s_2, \ldots$ satisfy the same second order linear homogeneous recurrence with constant coefficients, then for any constants α_1 and α_2, and for all $n \geq 0$, we have $a_n = \alpha_1 r_n + \alpha_2 s_n$, also satisfies the same recurrence.

What does it mean for our example earlier?

$$a_n = a_{n-1} + 2a_{n-2}$$

we do have base cases yet!

- 2^n and $(-1)^n$ are solutions we have seen this earlier.
- $4 \cdot 2^n + 5 \cdot (-1)^n$ is also a solution! So is $2^n + (-3) \cdot (-1)^n$.
- In fact, for any constants α_1 and α_2, $\alpha_1 \cdot 2^n + \alpha_2 \cdot (-1)^n$ is a solution.

Linear Homogeneous Recurrences of degree two with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

Claim: If $r_0, r_1, r_2, \ldots,$ and $s_0, s_1, s_2, \ldots$ satisfy the same second order linear homogeneous recurrence with constant coefficients, then for any constants α_1 and α_2, and for all $n \geq 0$, we have $a_n = \alpha_1 r_n + \alpha_2 s_n$, also satisfies the same recurrence.

Proof:

- Since $r_0, r_1, r_2 \ldots$ satisfies the recurrence, we know $r_n = c_1 r_{n-1} + c_2 r_{n-2}$
- Since $s_0, s_1, s_2 \ldots$ satisfies the recurrence, we know $s_n = c_1 s_{n-1} + c_2 s_{n-2}$

Let $a_0, a_1, a_2, \ldots$ be the sequence defined by $\alpha_1 r_n + \alpha_2 s_n$. Then,

$$\begin{aligned} c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} &= c_1(\alpha_1 r_{n-1} + \alpha_2 s_{n-1}) + c_2(\alpha_1 r_{n-2} + \alpha_2 s_{n-2}) \\ &= \alpha_1(c_1 \cdot r_{n-1} + c_2 \cdot r_{n-2}) + \alpha_2(c_1 \cdot s_{n-1} + c_2 \cdot s_{n-2}) \\ &= \alpha_1 r_n + \alpha_2 s_n = a_n \end{aligned}$$

This completes the proof.

Back to Example 1

$$a_n = a_{n-1} + 2a_{n-2}$$
$$a_0 = 1, \quad a_1 = 8$$

- 2^n and $(-1)^n$ are solutions (not satisfying base cases).
- Goal: Obtain a closed form for the recurrence including base cases.

We know that $\alpha_1 \cdot 2^n + \alpha_2 \cdot (-1)^n$ is a solution, for any constants α_1, α_2.

We use base cases to get values of α_1 and α_2.

$$a_0 = 1 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot (-1)^0 = \alpha_1 + \alpha_2$$
$$a_1 = 8 = \alpha_1 \cdot 2^1 + \alpha_2 \cdot (-1)^1 = 2 \cdot \alpha_1 - \alpha_2$$

Solving this for α_1, α_2 gives us : $\alpha_1 = 3$ and $\alpha_2 = -2$.

Verify that $\boxed{3 \cdot 2^n + (-2) \cdot (-1)^n}$ is a solution to the recurrence.

Characteristic equation

Let the recurrence be as follows where c_1 and c_2 are constants and $c_2 \neq 0$.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

Claim: The above recurrence is satisfied by the sequence

$$1, t, t^2, t^3, \ldots, t^n, \ldots$$

where t is a non-zero real number **iff** t satisfies

$$t^2 - c_1 t - c_2 = 0$$

Ex: Write down the proof for the above. Note that the proof has two parts.

$$\boxed{t^2 - c_1 t - c_2 = 0}$$

is called as the characteristic equation of the recurrence relation.

- The characteristic equation for Example 1 is $t^2 - t - 2 = 0$.
- 2 and (-1) are indeed solutions of the above equation.

Linear Homogeneous Recurrences of degree two with constant coefficients

Input:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$
$$a_0 = x \quad a_1 = y$$

Goal: To obtain a closed form satisfying base cases.

- Write down characteristic equation $t^2 - c_1 t - c_2 = 0$.
- Solve the characteristic equation to get roots.
  two possibilities – two distinct roots or a single root with multiplicity two

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Distinct Roots case: If two distinct roots, r_1 and r_2, then by previous claim, we know that following sequence also satisfies the recurrence

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

Use base cases $a_0 = x$ and $a_1 = y$ to compute values for α_1 and α_2.

Single Roots case: Coming up.

Example 2: Distinct roots case

Fibonacci Sequence

$$f_n = f_{n-1} + f_{n-2}$$
$$f_0 = 1 \quad f_1 = 1$$

We can obtain a closed form using the above technique.

- Characteristic equation: $t^2 - t - 1 = 0$.
- Roots of the characteristic equation are:

$$r_1 = \frac{1+\sqrt{5}}{2} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

- Solve $\alpha_1 r_1^0 + \alpha_2 r_2^0 = 1$ and $\alpha_1 r_1^1 + \alpha_2 r_2^1 = 1$ to obtain α_1 and α_2.
- Final solution is

$$f_n = \alpha_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Ex:

- Find values of α_1 and α_2.
- Change the base cases to say $f_0 = 2$ and $f_1 = 3$ and observe how the solution changes. Check for another choice of base cases.

Summary

- Special types of recurrences: Linear homogeneous recurrence relations of degree two with constant coefficients.
- Characteristic equation.
- Closed form when the characteristic equation has distinct roots.
- Upcoming: Single roots case and the non-homogeneous case.
- References: Section 8.2 [KR]