Advanced Counting Techniques

CS1200, CSE IIT Madras

Meghana Nasre

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Advanced Counting Techniques

- Principle of Inclusion-Exclusion ✓
- Recurrences and its applications ✓
- Solving Recurrences

Repeated Substitution Method : Learnings

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- An elementary method to solve recurrences.
  elementary does not mean simple, but a something that does not need background
- Need to observe a pattern. Do not oversimplify.
- Creativity and experience with summation of series help.
- However, the pattern has to be observed for each recurrence and there is no generic rule. Are there some recurrences that can be solved by a formula?

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Ex: Solve by repeated substitution

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\begin{aligned}
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&= 2\sqrt{T(n-1)} && \textit{otherwise}
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Sol: $T(n) = 8n + 12$

Linear recurrences

Linear Homogeneous Recurrences with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

$1 \leq i \leq k$, c_i is a real number and $c_k \neq 0$

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Examples:

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- $T(n) = T(n-1) + T(n-2)$ and $T(0) = 0, T(1) = 1$.

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Non Examples:

- $T(n) = nT(n-1)$ and $T(0) = 1$ does not have constant coefficients
- $T(n) = T(n-1) \cdot T(n-2)$ and $T(0) = 0$, $T(1) = 1$. not linear

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Ans: Yes it is possible.

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- 2^n and $(-1)^n$ are solutions we have seen this earlier.
- $4 \cdot 2^n + 5 \cdot (-1)^n$ is also a solution! So is $2^n + (-3) \cdot (-1)^n$.
- In fact, for any constants α_1 and α_2, $\alpha_1 \cdot 2^n + \alpha_2 \cdot (-1)^n$ is a solution.

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This completes the proof.

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Solving this for α_1, α_2 gives us : $\alpha_1 = 3$ and $\alpha_2 = -2$.

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Verify that $\boxed{3 \cdot 2^n + (-2) \cdot (-1)^n}$ is a solution to the recurrence.

Characteristic equation

Let the recurrence be as follows where c_1 and c_2 are constants and $c_2 \neq 0$.

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Claim: The above recurrence is satisfied by the sequence

$$1, t, t^2, t^3, \ldots, t^n, \ldots$$

where t is a non-zero real number **iff** t satisfies

$$t^2 - c_1 t - c_2 = 0$$

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- 2 and (-1) are indeed solutions of the above equation.

Linear Homogeneous Recurrences of degree two with constant coefficients

Input:

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\begin{aligned}
a_n &= c_1 a_{n-1} + c_2 a_{n-2} \\
a_0 &= x \quad a_1 = y
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Single Roots case: Coming up.

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Ex:

- Find values of α_1 and α_2.
- Change the base cases to say $f_0 = 2$ and $f_1 = 3$ and observe how the solution changes. Check for another choice of base cases.

Summary

- Special types of recurrences: Linear homogeneous recurrence relations of degree two with constant coefficients.

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- References: Section 8.2 [KR]