Advanced Counting Techniques

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April 14, 2020

Advanced Counting Techniques

- Principle of Inclusion-Exclusion ✓
- Recurrences and its applications ✓
- Solving Recurrences

Recap: Solving recurrences

We have seen

- Method of Repeated Substitution.
- Linear Homogeneous recurrence relations of degree two with constant coefficients.
- Use of characteristic equation to solve these recurrences.

Today:

- What if roots are not distinct?
- How does this generalize beyond degree two?
- Beyond degree two with repeated roots.

Revisiting Distinct Roots case

Claim: Let c_1 and c_2 be constants and let the equation $t^2 - c_1 t - c_2 = 0$ have two distinct roots r_1 and r_2. The sequence $a_0, a_1, a_2, \ldots$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff for constants α_1 and α_2, we have

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

- We used this claim to solve these special recurrences.
- We did not prove the claim. (see Theorem 1 Section 8.2 [KR])
- Check out where the theorem does not hold for the single roots case.

Example 1

$$a_n = 4a_{n-1} - 4a_{n-2}$$
$$a_0 = 1; \quad a_1 = 3$$

- The characteristic equation is $t^2 - 4t + 4 = 0$ with a single root $r = 2$.
- Observe that $1, 2, 2^2, 2^3, \ldots,$ is a sequence that satisfies the recurrence, but not the base cases.
- However, we need one more solution (even if it does not satisfy the base cases) to obtain the final solution.
- We observe that $0, 2, 8, 24, 64, \ldots,$ is also a sequence that satisfies the recurrence, again not the base cases.
  In fact the above sequence is $0 \cdot 2^0, 1 \cdot 2^1, 2 \cdot 2^2, \ldots, i \cdot 2^i, \ldots,$
- Now that we have two sequences 2^n and $n \cdot 2^n$ satisfying the recurrence relation (without base cases), we use our earlier idea.
  That is, $\alpha_1 2^n + \alpha_2 n \cdot 2^n$ satisfies the recurrence.
- Finally we use base cases to determine, $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2}$.

Qn: Does it hold in general?

Single Roots case

Claim: Let c_1 and c_2 be constants and the equation $t^2 - c_1 t - c_2 = 0$ have single root r. The sequence $a_0, a_1, a_2, \ldots$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff for constants α_1 and α_2, we have

$$a_n = \alpha_1 \cdot r^n + \alpha_2 \cdot n \cdot r^n$$

Proof (Part 1): Assume r is a single root of the characteristic equation. Let α_1 and α_2 be constants.

To prove: The sequence given by $a_n = \alpha_1 \cdot r^n + \alpha_2 \cdot n \cdot r^n$ satisfies the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2}$.

Since r is a root of $t^2 - c_1 t - c_2 = 0$, we have $r^2 = c_1 r + c_2$. Consider

$$\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1 \left(\alpha_1 r^{n-1} + \alpha_2 (n-1) r^{n-1} \right) + c_2 \left(\alpha_1 r^{n-2} + \alpha_2 (n-2) r^{n-2} \right) \\
&= \alpha_1 \left(c_1 r^{n-1} + c_2 r^{n-2} \right) + \alpha_2 \left(c_1 (n-1) r^{n-1} + c_2 (n-2) r^{n-2} \right) \\
&= \alpha_1 r^{n-2} (c_1 r + c_2) + \alpha_2 r^{n-2} (c_1 r (n-1) + c_2 (n-2)) \\
&= \alpha_1 r^{n-2} (c_1 r + c_2) + \alpha_2 r^{n-2} (c_1 r n + c_2 n - (c_1 r + 2 c_2)))
\end{aligned}$$

Single Roots case

Claim: Let c_1 and c_2 be constants and the equation $t^2 - c_1 t - c_2 = 0$ have single root r. The sequence $a_0, a_1, a_2, \ldots$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff for constants α_1 and α_2, we have

$$a_n = \alpha_1 \cdot r^n + \alpha_2 \cdot n \cdot r^n$$

Proof (Part 1 continued): Consider

$$\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1 \left(\alpha_1 r^{n-1} + \alpha_2 (n-1) r^{n-1}\right) + c_2 \left(\alpha_1 r^{n-2} + \alpha_2 (n-2) r^{n-2}\right) \\
&= \alpha_1 r^{n-2} (c_1 r + c_2) + \alpha_2 r^{n-2} (c_1 rn + c_2 n - (c_1 r + 2c_2))) \\
&= \alpha_1 r^{n-2} (c_1 r + c_2) + \alpha_2 r^{n-2} (c_1 rn + c_2 n) \\
&= \alpha_1 r^{n-2} \cdot r^2 + \alpha_2 r^{n-2} \cdot n \cdot r^2 = \alpha_1 r^n + \alpha_2 \cdot n \cdot r^n = a_n
\end{aligned}$$

Hence proved

Recall we are in the single roots case. Therefore,

$$c_1^2 + 4 \cdot c_2 = 0 \quad \text{and} \quad r = \frac{c_1}{2}$$

Use the fact that we are in the single roots case to show $c_1 r + 2c_2 = 0$.

Single Roots case

Claim: Let c_1 and c_2 be constants and the equation $t^2 - c_1 t - c_2 = 0$ have single root r. The sequence $a_0, a_1, a_2, \ldots$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff for constants α_1 and α_2, we have

$$a_n = \alpha_1 \cdot r^n + \alpha_2 \cdot n \cdot r^n$$

Proof (Part 2): We now prove that for every solution $a_0, a_1, \ldots$ to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ with base cases as $a_0 = x$ and $a_1 = y$, there exists constants α_1 and α_2 such that $a_n = \alpha_1 \cdot r^n + \alpha_2 \cdot n \cdot r^n$.

Using the base cases, we note that

$$\begin{aligned} x &= \alpha_1 r^0 + \alpha_2 \cdot 0 \cdot r^0 \implies \alpha_1 = x \\ y &= \alpha_1 r^1 + \alpha_2 \cdot 1 \cdot r^1 \implies \alpha_2 = \frac{y - x \cdot r}{r} \end{aligned}$$

Clearly, for this choice of constants $\alpha_1 r^n + \alpha_2 n r^n$ satisfies the base cases.

Now, we note that $a_0, a_1, a_2, \ldots$ is a solution to the recurrence (by assumption). Further, note that for a linear homogeneous recurrence relation of degree two with two base cases, has a unique solution.

We have already shown (in Part 1) that for some constants α_1 and α_2 the formula $\alpha_1 r^n + \alpha_2 n r^n$ satisfies the recurrence. Thus, the two sequences are the same and hence $a_n = c_1 a_{n-1} + c_2 a_{n-2} = \alpha_1 r^n + \alpha_2 n r^n$ for all $n \geq 0$.

Example 2

Ex: Consider the following finite recursive sequence containing $M + 1$ terms.

$$
\begin{aligned}
a_n &= 2a_{n-1} - a_{n-2} \\
a_0 &= 1; \quad a_M = 0
\end{aligned}
$$

- Note that the two values given are first and the last terms.
- Can you compute a closed form expression for the n-th term where $0 \leq n \leq M$?

Beyond Degree Two

Two natural questions

- What if degree is more than 2?
- What if multiple roots of different multiplicities?

Claim: Let $c_1, c_2, \ldots, c_k$ be constants. Suppose the characteristic equation $t^k - c_1 t^{k-1} - \ldots c_k = 0$ has k distinct roots, namely $r_1, r_2, \ldots, r_k$. Then the sequence $a_0, a_1, a_2, \ldots$ is a solution to the recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

iff for $n = 0, 1, 2,$ we have the following, where α_is are constants.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

We show the claim applied to an example.

Example 3

$$a_n = 4a_{n-2} + 4a_{n-3} - a_{n-1}$$
$$a_0 = 8 \quad a_1 = 6 \quad a_2 = 26$$

Ex: Apply the above claim and get the closed form expression.

Sol:

- Characteristic equation is: $t^3 + t^2 - 4t - 4 = 0$.
- Roots are $r_1 = -1, r_2 = -2, r_3 = 2$.
- Thus, $a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_3(2)^n$.
- Use bases cases to get $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 5$.

Beyond Degree Two with repeated roots

Claim: Let $c_1, c_2, \ldots, c_k$ be constants. Suppose the characteristic equation $t^k - c_1 t^{k-1} - \ldots - c_k = 0$ has ℓ distinct roots $r_1, r_2, \ldots, r_\ell$ with multiplicities $m_1, m_2, \ldots, m_\ell$ respectively. Then the sequence $a_0, a_1, a_2, \ldots$ is a solution to the recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

iff for $n = 0, 1, 2,$ we have the following, where $\alpha_{i,j}$ are constants.

$$\begin{aligned} a_n = \quad & (\alpha_{1,0} + \alpha_{1,1} \cdot n + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) \cdot r_1^n \\ + \quad & (\alpha_{2,0} + \alpha_{2,1} \cdot n + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) \cdot r_2^n \\ + \ldots + \quad & (\alpha_{\ell,0} + \alpha_{\ell,1} \cdot n + \ldots + \alpha_{\ell,m_\ell-1} n^{m_\ell-1}) \cdot r_n^\ell \end{aligned}$$

- Note how the degree two repeated case is a special case of the above.
- Next, we show the claim applied to an example.

Example 4

$$a_n = 8a_{n-2} - 16a_{n-4} \quad \text{for} n \geq 4$$
$$a_0 = 1 \quad a_1 = 4 \quad a_2 = 28 \quad a_3 = 32$$

Ex: Apply the above claim and get the closed form expression.

Sol:

- Characteristic equation is: $t^4 - 8t^2 + 16 = 0$.
- Roots are $r_1 = 2$ with $m_1 = 2$ and $r_2 = -2$ with $m_2 = 2$.
- Thus, $a_n = \alpha_1(2)^n + \alpha_2 n(2)^n + \alpha_3(-2)^n + \alpha_4 n(-2)^n$.
- Use bases cases to get $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 0$ and $\alpha_4 = 1$.

Summary

- Linear Homogeneous Recurrence Relations with constant coefficients.
- Degree two with repeated roots.
- Beyond Degree two and with repeated roots.
- Upcoming: Non-homogeneous case and use of Generating Functions.
- References: Section 8.2 [KR]