Advanced Counting Techniques

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Advanced Counting Techniques

- Principle of Inclusion-Exclusion ✓
- Recurrences and its applications ✓
- Solving Recurrences

Recap: Solving recurrences

We have seen

- Method of Repeated Substitution.
- Linear Homogeneous recurrence relations with constant coefficients.
- Use of characteristic equation to solve these recurrences.

Today:

- Non-homogeneous case.
- Comparing functions (a short detour).

Example 1

$$a_n = 3 \cdot n \cdot 2^n - a_{n-1}$$

- This is a non-homogeneous linear recurrence with constant coefficients. The presence of the term $F(n) = 3 \cdot n \cdot 2^n$ makes it non-homogeneous.
- The recurrence $a_n = -a_{n-1}$ is the *"associated homogeneous recurrence"*. We know the general form for the solution : call this solution $\{a_n^{(h)}\}$. For the example it is $\alpha_1 \cdot (-1)^n$.
- Suppose there is a solution to the non-homogeneous recurrence that we somehow guess!, lets call it $\{a_n^{(p)}\}$. Note that this may still not satisfy base cases. In the example, the formula $(2n + \frac{2}{3}) \cdot 2^n$ satisfies the given recurrence.

Then the solution to the given recurrence is of the form

$$\begin{aligned} \{a_n\} &= \{a_n^{(p)}\} + \{a_n^{(h)}\} \\ &= \left(2n + \frac{2}{3}\right) \cdot 2^n + \alpha_1(-1)^n \end{aligned}$$

If base case is $a_1 = 1$ we get $\alpha_1 = \frac{13}{3}$. Check out the formula works!

Example 1 continued

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  For the example it is $\alpha_1 \cdot (-1)^n$.
- Suppose there is a solution to the non-homogeneous recurrence that we somehow guess!, lets call it $\{a_n^{(p)}\}$. Note that this may still not satisfy base cases.
  In the example, the formula $(2n + \frac{2}{3}) \cdot 2^n$ satisfies the given recurrence.

Some unanswered questions:

- Why can we add the two solutions? That is, why is $\{a_n\} = \{a_n^{(p)}\} + \{a_n^{(h)}\}$?
- How did we guess $(2n + \frac{2}{3}) \cdot 2^n$?
- Does it depend on the function $F(n)$?

Non-Homogeneous Case

Why can we add the two solutions? That is, why is $\{a_n\} = \{a_n^{(p)}\} + \{a_n^{(h)}\}$?

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

- Let $\{a_n^{(p)}\}$ be some (guessed!) solution to the recurrence (not necessarily satisfying base cases).
- Let $\{b_n\}$ be another solution to the recurrence.
- The difference $\{b_n\} - \{a_n^{(p)}\}$ is a solution to the associated homogeneous recurrence relation.
  short justification: the term $F(n)$ cancels out in the difference.

See Theorem 5 and its proof in Section 8.2[KR].

Non-Homogeneous Case

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

- Let $F(n) = q(n) \cdot s^n$ where $q(n)$ is a polynomial of degree t and s is a constant.
  Note that this is a specific type of $F(n)$ that we are dealing with.
- Solve the associated homogeneous part to get a general form.
  Do not fix the constants $\alpha_1, \alpha_2, \ldots$.
  Let $r_1, r_2, \ldots, r_k$ be the roots of the charac. equation for the homogeneous part.

Two possibilities

Case 1: s is not one of the roots

Guess $\{a_n^{(p)}\} = (\beta_0 + \beta_1 n + \ldots + \beta_t n^t) \cdot s^n$

Obtain the values for these β_i, we show next how.

Case 2: s is one of the roots with multiplicity m

Guess $\{a_n^{(p)}\} = (\beta_0 + \beta_1 n + \ldots + \beta_t n^t) \cdot n^m s^n$

Obtain the values for these β_i, we show next how.

Back to Example 1

$$a_n = 3 \cdot n \cdot 2^n - a_{n-1}$$

- The associated homogeneous recurrence $a_n = -a_{n-1}$ has a solution $r_1 = -1$.
- $F(n) = 3 \cdot n \cdot 2^n$, thus, $q(n) = 3n$ and $s = 2$.
- Guess $\{a_n^{(p)}\} = (\beta_0 + \beta_1 n) \cdot 2^n$.

Since $\{a_n^{(p)}\}$ is a "guess" for the recurrence, we substitute it in the given recurrence.

$$\begin{aligned} a_n + a_{n-1} &= 3 \cdot n \cdot 2^n \\ (\beta_0 + \beta_1 n) \cdot 2^n + (\beta_0 + \beta_1(n-1)) \cdot 2^{n-1} &= 3 \cdot n \cdot 2^n \\ \frac{3}{2}\beta_1 n 2^n + \left(-\frac{1}{2}\beta_1 + \frac{3}{2}\beta_0\right) \cdot 2^n &= 3 \cdot n \cdot 2^n \end{aligned}$$

$$\implies \frac{3}{2}\beta_1 = 3 \quad \text{and} \quad \left(-\frac{1}{2}\beta_1 + \frac{3}{2}\beta_0\right) = 0$$

$$\implies \beta_1 = 2 \quad \text{and} \quad \beta_0 = \frac{2}{3}$$

Example 2

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} + 3n + 1 \\ a_0 &= 2; \quad a_1 = 3 \end{aligned}$$

- The associated homogeneous recurrence is familiar fibonacci sequence.

$$a_n^{(h)} = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

- $F(n) = 3n + 1 = (3n+1) \cdot 1^n$. Thus $q(n) = (3n+1)$ and $s = 1$.
- Hence we fall in Case 1 and guess $a_n^{(p)} = (\beta_0 + \beta_1 n) \cdot 1^n$.

$$\begin{aligned} (\beta_0 + \beta_1 n) \cdot 1^n &= (\beta_0 + \beta_1 (n-1)) \cdot 1^{n-1} + (\beta_0 + \beta_1(n-2)) \cdot 1^{n-2} + 3n + 1 \\ 0 &= (3 + \beta_1) n + (\beta_0 - 3\beta_1 + 1) \end{aligned}$$

- This gives us $\beta_0 = -10$ and $\beta_1 = -3$.

Thus,

$$a_n = -3n - 10 + \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Use base cases to get $\alpha_1 = 6 + 2\sqrt{5}$ and $\alpha_2 = 6 - 2\sqrt{5}$

Example 3

$$\begin{aligned} a_n &= 4a_{n-1} - 4a_{n-2} + (n+1) \cdot 2^n \\ a_0 &= 2; \quad a_1 = 3 \end{aligned}$$

- The associated homogeneous recurrence is $a_n = 4a_{n-1} + 4a_{n-2}$ whose characteristic equation has a single root $r_1 = 2$ of multiplicity 2.

$$a_n^{(h)} = \alpha_1 \cdot 2^n + \alpha_2 \cdot n \cdot 2^n$$

- $F(n) = (n+1) \cdot 2^n$, thus $q(n) = (n+1)$ and $s = 2$.
- Hence we fall in Case 2 and guess $a_n^{(p)} = (\beta_0 + \beta_1 n) \cdot n^2 \cdot 2^n$.
- Substitute it in the recurrence to obtain $\beta_0 = 1$ and $\beta_1 = \frac{1}{6}$.

Ex: Use base cases to compute α_1 and α_2.

Solving Recurrences: Summary

- Repeated substitution method (applicable in general).
- A special case of recurrence relations: Linear recurrences with constant coefficients.
  homogeneous and non-homogeneous cases.
- For the non-homogeneous case: dealt with restricted case of $F(n)$.
- Other methods: Master method (to deal with recurrences in divide and conquer algorithms), generating functions.
- Reference : Section 8.2 [KR]