Advanced Counting Techniques

CS1200, CSE IIT Madras

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Advanced Counting Techniques

- Principle of Inclusion-Exclusion ✓
- Recurrences and its applications ✓
- Solving Recurrences

Recap: Solving recurrences

We have seen

- Method of Repeated Substitution.
- Linear Homogeneous recurrence relations with constant coefficients.
- Use of characteristic equation to solve these recurrences.

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Today:

- Non-homogeneous case.
- Comparing functions (a short detour).

Example 1

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If base case is $a_1 = 1$ we get $\alpha_1 = \frac{13}{3}$. Check out the formula works!

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Some unanswered questions:

- Why can we add the two solutions? That is, why is $\{a_n\} = \{a_n^{(p)}\} + \{a_n^{(h)}\}$?
- How did we guess $(2n + \frac{2}{3}) \cdot 2^n$?
- Does it depend on the function $F(n)$?

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- The difference $\{b_n\} - \{a_n^{(p)}\}$ is a solution to the associated homogeneous recurrence relation.

  short justification: the term $F(n)$ cancels out in the difference.

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See Theorem 5 and its proof in Section 8.2[KR].

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- Solve the associated homogeneous part to get a general form.
  Do not fix the constants $\alpha_1, \alpha_2, \ldots$.
  Let $r_1, r_2, \ldots, r_k$ be the roots of the charac. equation for the homogeneous part.

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Case 1: s is not one of the roots

Guess $\{a_n^{(p)}\} =$
$(\beta_0 + \beta_1 n + \ldots + \beta_t n^t) \cdot s^n$

Obtain the values for these β_i, we show next how.

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Case 2: s is one of the roots with multiplicity m

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$$\implies \frac{3}{2}\beta_1 = 3 \quad \text{and} \quad \left(-\frac{1}{2}\beta_1 + \frac{3}{2}\beta_0\right) = 0$$

$$\implies \beta_1 = 2 \quad \text{and} \quad \beta_0 = \frac{2}{3}$$

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Thus,

$$a_n = -3n - 10 + \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Use base cases to get $\alpha_1 = 6 + 2\sqrt{5}$ and $\alpha_2 = 6 - 2\sqrt{5}$

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- Hence we fall in Case 2 and guess $a_n^{(p)} = (\beta_0 + \beta_1 n) \cdot n^2 \cdot 2^n$.
- Substitute it in the recurrence to obtain $\beta_0 = 1$ and $\beta_1 = \frac{1}{6}$.

Ex: Use base cases to compute α_1 and α_2.

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  homogeneous and non-homogeneous cases.
- For the non-homogeneous case: dealt with restricted case of $F(n)$.
- Other methods: Master method (to deal with recurrences in divide and conquer algorithms), generating functions.

Solving Recurrences: Summary

- Repeated substitution method (applicable in general).
- A special case of recurrence relations: Linear recurrences with constant coefficients.
  homogeneous and non-homogeneous cases.
- For the non-homogeneous case: dealt with restricted case of $F(n)$.
- Other methods: Master method (to deal with recurrences in divide and conquer algorithms), generating functions.
- Reference : Section 8.2 [KR]