Structured Sets

CS1200, CSE IIT Madras

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Structured Sets

- Relational Structures
  - Properties and closures
  - Equivalence Relations
  - Partially Ordered Sets (Posets) and Lattices
- Algebraic Structures
  - Groups and Rings

Recap: Binary relations and properties

A binary relation R on a set A is a subset of the Cartesian product $A \times A$.

Properties of Binary Relations

- Reflexive: If for every $a \in A$, $(a, a) \in R$.
  - $\leq$ on Z^+, $\geq$ on Z^+.
- Symmetric: If $(a, b) \in R \rightarrow (b, a) \in R$, for all $a, b \in A$
  - $=$ on Z^+
  - "is a cousin of" on the set of people.
- Antisymmetric: If $((a, b) \in R$ and $(b, a) \in R) \rightarrow a = b$, for all $a, b \in A$.
  - $\leq$ on Z^+ , $\geq$ on Z^+.
- Transitive: If for all $a, b, c \in A$, $((a, b) \in R$ and $(b, c) \in R) \rightarrow (a, c) \in R$.
  - "is an ancestor of" on the set of people.

Recall representation of relation using matrices.

Closure w.r.t. a Property

Let R be a relation on set A and let $\mathcal{P}$ be some property.

Think of $\mathcal{P}$ as one of reflexive, transitive and so on

Note that R may or may not posses $\mathcal{P}$.

Closure w.r.t. $\mathcal{P}$: If there exists a relation S (on A) with the property $\mathcal{P}$ containing R such that S is a subset of **every relation** with property $\mathcal{P}$ **containing** R, then S is called the closure of R w.r.t. $\mathcal{P}$.

Note that:

- S must contain R.
- S may not exist. Ex: Think of a property for which this happens.

Example: $\mathcal{P}$ is reflexivity.

$$A = \{1, 2, 3\} \qquad R = \{(1, 1), (1, 2), (1, 3)\}.$$

- Clearly R is not reflexive.
- $S_1 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ is not a closure (S_1 does not contain R).
- $S_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3)\}$ is a closure of R.
- $S_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (3, 1)\}$ is not a closure.
  (S_3 is not a subset of S_2 which satisfies reflexivity and contains R.)

Closure w.r.t. a Property

Let R be a relation on set A and let $\mathcal{P}$ be some property.

Closure w.r.t. $\mathcal{P}$: If there exists a relation S (on A) with the property $\mathcal{P}$ such that S is a subset of **every relation** with property $\mathcal{P}$ **containing** R, then S is called the closure of R w.r.t. $\mathcal{P}$.

Reflexive closure: Add to R all the "diagonal elements". That is,

$$S = R \cup \{(a, a) \mid a \in A\}$$

Symmetric closure: Add to R the "inverse relation". That is,

$$S = R \cup \{(b, a) \mid (a, b) \in R\}$$

- Make sure that in both cases it is indeed the closure.
- What about transitive closure? coming up.

Representing a relation: matrix and graph

$$A = \{x, y, z, w\} \text{ and } R = \{(x, y), (y, z), (z, w)\}$$

Matrix Representation: Entry $[a, b] = 1$ iff $(a, b) \in R$.

$$\begin{array}{c c}
 & \begin{array}{cccc} x & y & z & w \end{array} \\
\begin{array}{c} x \\ y \\ z \\ w \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{array}$$

Graph Representation:

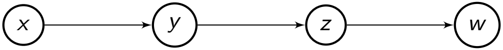
A node / vertex for every element $a \in A$. An edge from a to b iff $(a, b) \in R$.

$$\begin{tikzcd} x \arrow[r] & y \arrow[r] & z \arrow[r] & w \end{tikzcd}$$

Representing a relation: matrix and graph

$$A = \{x, y, z, w\} \text{ and } R = \{(x, y), (y, z), (z, w)\}$$

Graph Representation:
A node / vertex for every element $a \in A$. An edge from a to b iff $(a, b) \in R$.



Path in a graph: Sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k)$

Here k is the number of edges in the path, which is equal to the length of the path.
Recall Goal: To compute transitive closure.

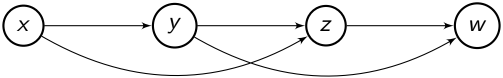
Transitive Closure: First attempt

$$A = \{x, y, z, w\} \text{ and } R = \{(x, y), (y, z), (z, w)\}$$

Algo-1:

- If R is transitive, we are done, return $S = R$.
- Else $S = R$ and for every a, b, c $\in A$, if (a, b) and (b, c) belong to R then add (a, c) to S.

In our example, it implies $S = R \cup \{(x, z), (y, w)\}$.



However S is not transitive! (x, z) and (z, w) is present but (x, w) is absent!

Apply Algo-1 on S? Does it give a transitive relation? How long do we do this?

The Matrix way

$$A = \{x, y, z, w\} \text{ and } R = \{(x, y), (y, z), (z, w)\}$$

Algo-1:

- If R is transitive, we are done, return $S = R$.
- Else $S = R$ and for every a, b, c $\in A$, if (a, b) and (b, c) belong to R then add (a, c) to S.

The above is equivalent to $S = R \cup R^2$.

$$M^2 = \begin{array}{c c} & \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{array}{c} x \\ y \\ z \\ w \end{array} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad M \vee M^2 = \begin{array}{c c} & \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{array}{c} x \\ y \\ z \\ w \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- We know S is not transitive.
- What does M^2 represent in terms of paths? What does $M \vee M^2$ represent? R^2 is the set of pairs (a, b) such that $(a, c) \in R$ and $(c, b) \in R$, equivalently, there is a two length path from a to b in the graph.

The Matrix way

$$A = \{x, y, z, w\} \text{ and } R = \{(x, y), (y, z), (z, w)\}$$

- R^2 is the set of pairs (a, b) such that $(a, c) \in R$ and $(c, b) \in R$.
- R^3 is the set of pairs (a, b) such that $(a, c) \in R$ and $(c, d) \in R$ and $(d, b) \in R$. Note that we do not claim that a, b, c, d are distinct!
- R^n is the set of pairs (a, b) such that there exists $x_1, \ldots, x_{n-1}$ where $(a, x_1) \in R$ and $(x_2, x_2) \in R$ and $\ldots$ and $(x_{n-1}, b) \in R$.

$$R^* = R \cup R^2 \cup \ldots \cup R^n$$

Thus R^* denotes pairs (a, b) such that either

- there is a direct path from a to b, or
- there is a path from a to c and then c to b, or, ..
- there is a path from a to x_1 and x_1 to x_2 and $\ldots x_{n-1}$ to b.

This captures only n length paths. What if there are longer paths?

Paths in the associated graph

Claim: Let A be a set on n elements and R be a relation on A. If the graph contains a path of length at least one from a to b then there is also a path of length at most n from a to b.

Proof Sketch: If there is "long" path exceeding n edges from a to b, consider the shortest path from a to b.

If the shortest path contains at most n edges, we are done, else we will show a "shorter path" than the shortest path.

By pigeon hole principle, there will a vertex that repeats itself on the shortest path, thus creating a loop.

We can " short circuit" the loop and create a shorter path, a contradiction.

We can therefore consider only paths of length at most n.

Back to our goal: transitive closure

$$A = \{x, y, z, w\} \text{ and } R = \{(x, y), (y, z), (z, w)\}$$

- R^2 is the set of pairs (a, b) such that $(a, c) \in R$ and $(c, b) \in R$.
- R^3 is the set of pairs (a, b) such that $(a, c) \in R$ and $(c, d) \in R$ and $(d, b) \in R$. Note that we do not claim that a, b, c, d are distinct!
- R^n is the set of pairs (a, b) such that there exists $x_1, \ldots, x_{n-1}$ where $(a, x_1) \in R$ and $(x_2, x_2) \in R$ and $\ldots$ and $(x_{n-1}, b) \in R$.

$$\begin{aligned} R^* &= R \cup R^2 \cup \ldots \cup R^n \\ R^* &= \{(a, b) | \text{ there is a path from } a \text{ to } b \text{ in the graph corr. to } R\ \} \end{aligned}$$

Claim: R^* is the transitive closure of R.

Need to prove:

- R^* contains R ✓ (the way R^* is constructed.)
- R^* is transitive.
- R^* is a subset of any transitive relation S which contains R.

Back to our goal: transitive closure

$$A = \{x, y, z, w\} \text{ and } R = \{(x, y), (y, z), (z, w)\}$$

$$\begin{aligned} R^* &= R \cup R^2 \cup \ldots \cup R^n \\ R^* &= \{(a, b)| \text{ there is a path from } a \text{ to } b \text{ in the graph corr. to } R\ \} \end{aligned}$$

Sub-Claim: R^* is transitive.

Proof: Let (a, b) and (b, c) belong to R^*.

- Since $(a, b) \in R^*$ there is a path (of some length) from a to b in the graph corr. to R.
- Same holds for b to c.
- Combining the two paths we get a path from a to c in the graph corr. R. Thus, $(a, c) \in R^*$.

Hence R^* is transitive.

Back to our goal: transitive closure

$$A = \{x, y, z, w\} \text{ and } R = \{(x, y), (y, z), (z, w)\}$$

$$R^* = R \cup R^2 \cup \ldots \cup R^n$$
$$R^* = \{(a, b) |\text{ there is a path from } a \text{ to } b \text{ in the graph corr. to } R\ \}$$

Sub-Claim: R^* is a subset of every transitive relation S containing R.

Proof: Let S be some transitive relation containing R.

- Fact, to be verified: If S is any transitive relation, then $S^i \subseteq S$, for $i = 2, 3, \ldots, n$.
- Thus $S^* = S \cup S^2 \cup \ldots \cup S^n \subseteq S$.
- Finally, since $R \subseteq S$, we can claim that $R^* \subseteq S^*$ (because a path present in graph corr. to R is also present in S).
- Thus, R^* is contained in S^* which is contained in S.

This completes the proof.

Computing the transitive closure

$$A = \{x, y, z, w\} \text{ and } R = \{(x, y), (y, z), (z, w)\}$$

$$\begin{aligned} R^* &= R \cup R^2 \cup \ldots \cup R^n \\ R^* &= \{(a, b) | \text{ there is a path from } a \text{ to } b \text{ in the graph corr. to } R\ \} \end{aligned}$$

$$\begin{aligned} M^* &= M \vee M^2 \vee \ldots \vee M^n \\ M^* &= \{M^*[a, b] = 1 | \text{ there is a path from } a \text{ to } b \text{ in the graph corr. to } R\ \} \end{aligned}$$

Algo-2:

- Initialize: currM $=$ M, outM $=$ M
- For $i = 2$ to n
  - currM $=$ currM $\cdot M$
  - outM $=$ outM $\vee$ currM
- Return outM // this is the matrix for R^*

Summary

- Defined closure of w.r.t. a property of a relation.
- An algorithm to find transitive closure.
- Ex: Find out how many operations (multiplications and additions) are needed to compute transitive closure for a relation R on a set with n elements.
- Can we improve this?
- Reference: Section 9.4[KR]