Structured Sets

CS1200, CSE IIT Madras

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Structured Sets

- Relational Structures
  - Properties and closures
  - Equivalence Relations
  - Partially Ordered Sets (Posets) and Lattices
- Algebraic Structures
  - Groups and Rings

Recap: Binary relations and properties

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Recall representation of relation using matrices.

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- $S_3 = \{(1,1),(2,2),(3,3),(1,2),(1,3),(3,1)\}$ is not a closure.
  (S_3 is not a subset of S_2 which satisfies reflexivity and contains R.)

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- What about transitive closure? coming up.

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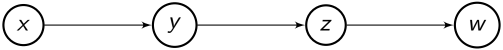

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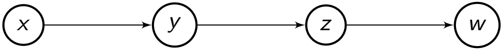

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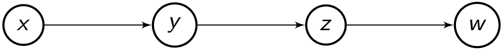

Path in a graph: Sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k)$

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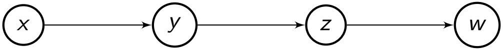

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Recall Goal: To compute transitive closure.

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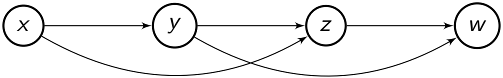

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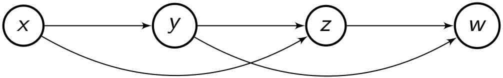

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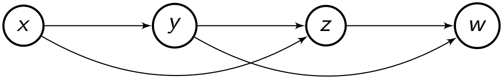

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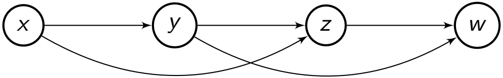

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Apply Algo-1 on S?

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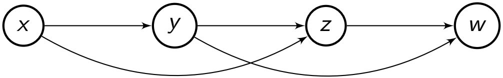


However S is not transitive! (x, z) and (z, w) is present but (x, w) is absent!

Apply Algo-1 on S? Does it give a transitive relation?

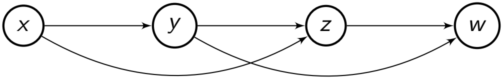
Transitive Closure: First attempt

$$A = \{x, y, z, w\} \text{ and } R = \{(x, y), (y, z), (z, w)\}$$

Algo-1:

- If R is transitive, we are done, return $S = R$.
- Else $S = R$ and for every a, b, c $\in A$, if (a, b) and (b, c) belong to R then add (a, c) to S.

In our example, it implies $S = R \cup \{(x, z), (y, w)\}$.



However S is not transitive! (x, z) and (z, w) is present but (x, w) is absent!

Apply Algo-1 on S? Does it give a transitive relation? How long do we do this?

The Matrix way

$A = \{x, y, z, w\}$ and $R = \{(x, y), (y, z), (z, w)\}$

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The above is equivalent to $S = R \cup R^2$.

$$M^2 = \begin{array}{c} \\ x \\ y \\ z \\ w \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

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- We know S is not transitive.
- What does M^2 represent in terms of paths?

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- We know S is not transitive.
- What does M^2 represent in terms of paths? What does $M \vee M^2$ represent? R^2 is the set of pairs (a, b) such that $(a, c) \in R$ and $(c, b) \in R$, equivalently, there is a two length path from a to b in the graph.

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$$R^* = R \cup R^2 \cup \ldots \cup R^n$$

Thus R^* denotes pairs (a, b) such that either

- there is a direct path from a to b, or
- there is a path from a to c and then c to b, or, ..
- there is a path from a to x_1 and x_1 to x_2 and $\ldots x_{n-1}$ to b.

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- there is a path from a to x_1 and x_1 to x_2 and $\ldots x_{n-1}$ to b.

This captures only n length paths. What if there are longer paths?

Paths in the associated graph

Claim: Let A be a set on n elements and R be a relation on A. If the graph contains a path of length at least one from a to b then there is also a path of length at most n from a to b.

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If the shortest path contains at most n edges, we are done, else we will show a "shorter path" than the shortest path.

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If the shortest path contains at most n edges, we are done, else we will show a "shorter path" than the shortest path.

By pigeon hole principle, there will a vertex that repeats itself on the shortest path, thus creating a loop.

We can " short circuit" the loop and create a shorter path, a contradiction.

We can therefore consider only paths of length at most n.

Back to our goal: transitive closure

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$$R^* = \{(a, b) | \text{ there is a path from } a \text{ to } b \text{ in the graph corr. to } R\ \}$$

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Claim: R^* is the transitive closure of R.

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Need to prove:

- R^* contains R

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Claim: R^* is the transitive closure of R.

Need to prove:

- R^* contains R ✓ (the way R^* is constructed.)

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- R^* contains R ✓ (the way R^* is constructed.)
- R^* is transitive.

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Claim: R^* is the transitive closure of R.

Need to prove:

- R^* contains R ✓ (the way R^* is constructed.)
- R^* is transitive.
- R^* is a subset of any transitive relation S which contains R.

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Sub-Claim: R^* is transitive.

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Sub-Claim: R^* is transitive.

Proof: Let (a, b) and (b, c) belong to R^*.

Back to our goal: transitive closure

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Sub-Claim: R^* is transitive.

Proof: Let (a, b) and (b, c) belong to R^*.

- Since $(a, b) \in R^*$ there is a path (of some length) from a to b in the graph corr. to R.

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- Same holds for b to c.

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Sub-Claim: R^* is transitive.

Proof: Let (a, b) and (b, c) belong to R^*.

- Since $(a, b) \in R^*$ there is a path (of some length) from a to b in the graph corr. to R.
- Same holds for b to c.
- Combining the two paths we get a path from a to c in the graph corr. R. Thus, $(a, c) \in R^*$.

Hence R^* is transitive.

Back to our goal: transitive closure

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- Fact, to be verified: If S is any transitive relation, then $S^i \subseteq S$, for $i = 2, 3, \ldots, n$.

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- Thus $S^* = S \cup S^2 \cup \ldots \cup S^n \subseteq S$.
- Finally, since $R \subseteq S$, we can claim that $R^* \subseteq S^*$ (because a path present in graph corr. to R is also present in S).
- Thus, R^* is contained in S^* which is contained in S.

This completes the proof.

Computing the transitive closure

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$$M^* = M \vee M^2 \vee \ldots \vee M^n$$

$$M^* = \{M^*[a, b] = 1 | \text{ there is a path from } a \text{ to } b \text{ in the graph corr. to } R\}$$

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Algo-2:

- Initialize: currM $= M$, outM $= M$
- For $i = 2$ to n
  - currM = currM $\cdot M$
  - outM = outM $\vee$ currM
- Return outM // this is the matrix for R^*

Summary

- Defined closure of w.r.t. a property of a relation.
- An algorithm to find transitive closure.
- Ex: Find out how many operations (multiplications and additions) are needed to compute transitive closure for a relation R on a set with n elements.
- Can we improve this?
- Reference: Section 9.4[KR]