Counting

CS1200, CSE IIT Madras

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Counting (without counting)

MATHEMATICS OF CHOICE
HOW TO COUNT WITHOUT COUNTING
IVAN NIVEN

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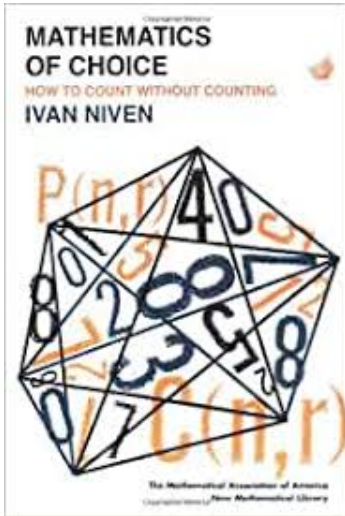
- Basic Counting Techniques ✓
- Pigeon Hole Principle (revisited) ✓
- Permutations and Combinations ✓
- Combinatorial Identities ✓
- Permutations and Combinations (revisited)

Permutations and Combinations with repetitions

Study the following questions.

- What is the number of length 10 strings formed using the upper case English alphabet?
- In how many ways can we order the letters of the word MISSISSIPPI to obtain distinguishable orderings?
- How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if each x_i is a non-negative integer.
- How many triples (i, j, k) can be formed where $1 \leq i \leq j \leq k \leq n$ where n is a positive integer?

The above examples have repetitions. However, techniques seen earlier do not take care of repetitions.

Permutations revisited

We consider three different scenarios.

Case 1: Input: n distinct objects   Number of r perm.: $P(n, r)$

Note that it is implicit that we do not repeat the input object in the permutation. For instance, if A, B, C, D is the input, the 3-permutation AAC is not valid (unless explicitly specified).

Case 2: Input: n distinct objects   Number of r perm. with repetitions: n^r

Now note that for the input A, B, C, D the 3-permutation AAC is valid.

Case 3: Input: n objects (not necessarily distinct) containing:

- n_1 indistinguishable obj. of type-1
- n_2 indistinguishable obj. of type-2
  ⋮
- n_k indistinguishable obj. of type-k.

Ex: MISSISSIPPI
Goal: Obtain distinguishable permutations
IISSIISSPMP and PIMISSSSIIP are distinct.

Permutations revisited

Case 3: Input: n objects (not necessarily distinct) containing n_1 indistinguishable obj. of type-1, n_2 indistinguishable obj. of type-2, ..., n_k indistinguishable obj. of type-k.
Goal: Count the number of distinguishable permutations

First note that n_1 objects can be placed in n locations in $\binom{n}{n_1}$ ways. This leaves $n - n_1$ locations for the remaining objects.

Now, n_2 objects can be placed in $n - n_1$ locations in $\binom{n-n_1}{n_2}$ ways.

Finally, n_k objects can be placed in $n - n_1 - n_2 - \cdots - n_{k-1}$ locations in $\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$ ways.

Thus total number of distinct permutations is

$$
= \binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdots \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}
$$

$$
= \frac{n!}{n_1! n_2! \cdots n_k!}
$$

Qn: Does it matter how we order the types? Check out!

- What is the number of length 10 strings formed using the upper case English alphabets?
  Ans: 26^{10} (Case 2)

- In how many ways can we order the letters of the word MISSISSIPPI to obtain distinguishable orderings?
  Ans: $\frac{11!}{4! \cdot 4! \cdot 2! \cdot 1!}$ (Case 3)

Combinations with repetitions

We know that the number of 3-sized subsets of $\{1, 2, 3, 4\}$ are exactly 4.

Now our goal is to count 3 sized sets with repetitions allowed. That is, we allow a choice $[1, 1, 2]$. However, this choice is the same as the choice $[1, 2, 1]$.

That is, (as earlier) order **does not** matter!

Goal: Count number of such 3-sized sets.
Ex: Try explicit enumeration and write down the answer.

We convert this into counting certain patterns.

- We have four categories to select from. We use 3 vertical bars to represent these categories.

$$| \quad | \quad |$$

- To select one item from that category we use $\times$ symbol in that category.
- We represent $[1, 1, 2]$ as:

$$\times \times \; | \; \times \; | \quad |$$

Combinations with repetitions

Our goal is to count 3 sized sets of $\{1, 2, 3, 4\}$ with repetitions allowed. That is, we allow a choice $[1, 1, 2]$. However, this choice is the same as the choice $[1, 2, 1]$.

That is, (as earlier) order **does not** matter!
We convert this into counting certain patterns.

- We represent $[1, 1, 2]$ as:

$$\times \times \mid \times \mid \quad \mid$$

- We represent $[4, 4, 4]$ as:

$$\mid \quad \mid \quad \mid \times \times \times$$

- What does this represent?

$$\mid \quad \mid \times \times \mid \times$$

Thus, every valid 3 sized set with repetition corresponds to a string of three vertical bars and three cross symbols!

Combinations with repetitions

Our goal is to count 3 sized sets of $\{1, 2, 3, 4\}$ with repetitions allowed. That is, we allow a choice $[1, 1, 2]$. However, this choice is the same as the choice $[1, 2, 1]$.

That is, (as earlier) order **does not** matter!
We convert this into counting certain patterns.

Every valid 3 sized set with repetition corresponds to a string of three vertical bars and three cross symbols!

Thus the total number of ways is to select three positions out of six (since once we select the position of the vertical bars out of the six positions, the position of the cross symbols is fixed!)

This is exactly equal to $\binom{6}{3}$.

Combinations with repetitions

Input: A set containing n distinct elements.

Goal: Number of r combinations with repetitions allowed.

Equivalent to number of ways of selecting r elements from n categories with repetitions allowed.

- Have $n - 1$ vertical bars to represent categories.
- Have r cross marks to represent the elements.
- Number of selections is equal to selecting r cross marks from $n - 1 + r$ positions.

That is, $\binom{n-1+r}{r} = \binom{n-1+r}{n-1}$.

Example: Selecting items with repetition from fixed categories

Qn: In a party, Sameer wants to set out 15 assorted juice packs. He has to select from 5 different flavours.

- How many different selections of 15 juice packs can Sameer set up?
- If he decides that his selection must contain at least 6 mango juice packs, how many different selections can he make?

Work out your answer before looking at the solution

Ans:

- The first one is an application of the formula we have seen before. We let 4 vertical bars denote the categories and 15 cross marks denote the juice packs. Thus Sameer has $\binom{4+15}{15}$ ways to select the assortment.
- Since 6 juice packs are already fixed, the goal is now to select only 9 packs. Note however, that we are allowed to select further mango juice packs, hence the number of categories still remains 5. Thus the number of ways possible for Sameer are $\binom{4+9}{9}$.

Example: Number of integral solutions of an equation

Qn: How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if each x_i is a non-negative integer? Work out your answer before looking at the solution

Ans: What are the categories? How many elements do we select?

- We consider dividing 10 as 1 unit each and selecting one unit from each of the 4 categories $x_1, \ldots x_4$.

- For example, a solution like $x_1 = 2, x_2 = 5, x_3 = 0$ and $x_4 = 3$ can be represented as

$$\times \times \ \big| \ \times \times \times \times \times \ \big| \ \ \big| \ \times \times \times$$

Thus the number solutions for the above equation are $\binom{3+10}{10}$.

What if there was an additional constraint that all x_is are at least 1?

Replace variables such that new variables are required to be only non-negative. However, the equation will change.

What if there were constraints of the form $0 \leq x_1 \leq 3$, $0 \leq x_2 \leq 4$ and so on?
Can you use the replacement idea above?

- Permutations with repetitions.
- Input with repetitions and counting distinct permutations.
- Combinations with repetitions.
- References: Section 6.5 [KR]