Counting

CS1200, CSE IIT Madras

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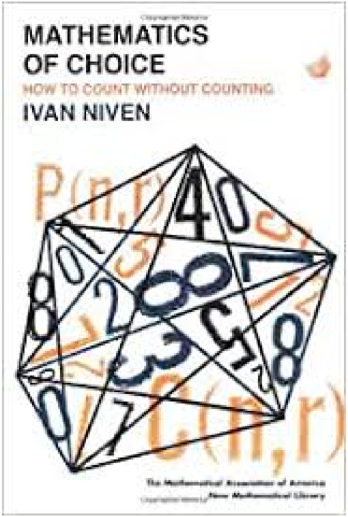
Counting (without counting)



- Basic Counting Techniques ✓
- Pigeon Hole Principle (revisited) ✓
- Permutations and Combinations ✓
- Combinatorial Identities ✓
- Permutations and Combinations (revisited)

Permutations and Combinations with repetitions

Study the following questions.

- What is the number of length 10 strings formed using the upper case English alphabet?
- In how many ways can we order the letters of the word MISSISSIPPI to obtain distinguishable orderings?
- How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if each x_i is a non-negative integer.
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The above examples have repetitions. However, techniques seen earlier do not take care of repetitions.

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Qn: Does it matter how we order the types? Check out!

Revisiting the questions posed

- What is the number of length 10 strings formed using the upper case English alphabets?
  Ans: 26^{10} (Case 2)
- In how many ways can we order the letters of the word MISSISSIPPI to obtain distinguishable orderings?
  Ans: $\frac{11!}{4! \cdot 4! \cdot 2! \cdot 1!}$ (Case 3)

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- To select one item from that category we use $\times$ symbol in that category.
- We represent $[1, 1, 2]$ as:

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Thus, every valid 3 sized set with repetition corresponds to a string of three vertical bars and three cross symbols!

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Every valid 3 sized set with repetition corresponds to a string of three vertical bars and three cross symbols!

Thus the total number of ways is to select three positions out of six (since once we select the position of the vertical bars out of the six positions, the position of the cross symbols is fixed!)

This is exactly equal to $\binom{6}{3}$.

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Goal: Number of r combinations with repetitions allowed.

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That is, $\binom{n-1+r}{r} = \binom{n-1+r}{n-1}$.

Example: Selecting items with repetition from fixed categories

Qn: In a party, Sameer wants to set out 15 assorted juice packs. He has to select from 5 different flavours.

- How many different selections of 15 juice packs can Sameer set up?
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- For example, a solution like $x_1 = 2, x_2 = 5, x_3 = 0$ and $x_4 = 3$ can be represented as

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Summary

- Permutations with repetitions.
- Input with repetitions and counting distinct permutations.
- Combinations with repetitions.
- References: Section 6.5 [KR]