Structured Sets

CS1200, CSE IIT Madras

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Structured Sets

- Relational Structures
  - Properties and closures ✓
  - Equivalence Relations
  - Partially Ordered Sets (Posets) and Lattices
- Algebraic Structures
  - Groups and Rings

Recap: Binary relations and properties

A binary relation R on a set S is a subset of the Cartesian product $S \times S$.

Properties of Binary Relations

- Reflexive: If for every $a \in S$, $(a, a) \in R$.
  - $\leq$ on Z^+, $\geq$ on Z^+.
- Symmetric: If $(a, b) \in R \rightarrow (b, a) \in R$, for all $a, b \in S$
  - $=$ on Z^+
  - "is a cousin of" on the set of people.
- Antisymmetric: If $((a, b) \in R$ and $(b, a) \in R) \rightarrow a = b$, for all $a, b \in S$.
  - $\leq$ on Z^+ , $\geq$ on Z^+.
- Transitive: If for all $a, b, c \in S$, $((a, b) \in R$ and $(b, c) \in R) \rightarrow (a, c) \in R$.
  - "is an ancestor of" on the set of people.

Equivalence Relations

If R on set S is

- reflexive, and
- symmetric, and
- transitive,

R is an <u>equivalence</u> relation.

Examples:

- "=" on Z^+
- $(a, b) \in R$ if 3 divides $(a - b)$.
- A : binary strings; $(s_1, s_2) \in R$ if first 10 bits of s_1 match with s_2.

- $(a, b) \in R$ implies a and b are equivalent.

Not equivalence relation:

- $\leq$ on Z^+.
- "divides" on Z^+.

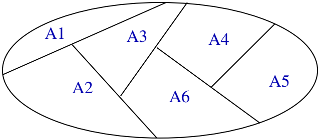
Equivalence relations

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

- $R = \{\ (a, b)\ \mid\ 3\ \text{divides}\ (a - b)\}$.
- $[a]$ denotes the set of elements $b \in S$ (in this case Z) such that $(a, b) \in R$.
- $[0] = \{\ a \in Z\ \mid\ 3\ \text{divides}\ (a - 0)\}$.
- $[0] = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$
- $[1] = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$
- $[2] = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$

Any equivalence relation R on S partitions the set S

Partition of a set S



A <u>partition</u> of a set S is a disjoint collection of subsets $A_1, A_2, \ldots, A_k$ such that

- $A_j \cap A_j = \phi$ for $i \neq j$.
- $\cup_{i=1}^{k} A_i = S$.

For an equivalence relation R on a set S, the following are equivalent.

(i) $(a, b) \in R$

(ii) $[a] = [b]$; $[a]$ denotes the class of $[a]$

(iii) $[a] \cap [b] \neq \emptyset$

Partition of a set S

For an equivalence relation R on a set S, the following are equivalent.

(i) $(a, b) \in R$ (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Proof: To show that (i) $\rightarrow$ (ii).

- Let $c \in [a]$. This implies $(a, c) \in R$ (by definition of $[a]$). Further $(c, a) \in R$, (by symmetry of R). Thus, $(c, b) \in R$ (by transitivity of R). Again applying symmetry $(b, c) \in R$. Thus $c \in [b]$. This concludes that $[a] \subseteq [b]$. A similar argument can be used to show $[b] \subseteq [a]$.

To show that (ii) $\rightarrow$ (iii). This holds because of reflexive property. We know $a \in [a]$. Thus, $a \in [a] \cap [b]$.

To show that (iii) $\rightarrow$ (i).

- Since $[a] \cap [b]$ is non-empty, we know that some $c \in [a]$ and $c \in [b]$. Thus, $(a, c) \in R$ and $(b, c) \in R$. By symmetry, $(c, b) \in R$. Together with transitivity of R, we have $(a, b) \in R$.

Observe how all three properties (reflexive, symmetry and transitivity) are used in the proof.

Equivalence relations

- Every equivalence relation partitions the set.
- Every partition of the set defines an equivalence relation.

Useful abstraction when we are interested in properties of the "classes" rather than individual elements.

- Set Z, $[0] = \{x \in Z \mid x \mod 3 = 0\}$, [1] and [2] defined appropriately.

Back to relations with properties

- S_1 – all words in English dictionary.
- Relation R_1 on S_1:
  - $(w_1, w_2) \in R_1$ if $w_1 = w_2$ or w_1 appears before w_2 in dictionary.

- S_2 – all subsets of $\{a, b, c\}$.
- Relation R_2 on S_2:
  - $(X, Y) \in R_2$ if $X \subseteq Y$.

- What properties do R_1 and R_2 satisfy?

Defn: If R on set S is reflexive, and anti-symmetric, and transitive, then R is a partial ordering on set S. Set S along with R is known as a partially ordered set or poset.
$a \preceq b$ is used to denote $(a, b) \in R$ when R is reflexive, anti-symmetric and transitive.

Examples:

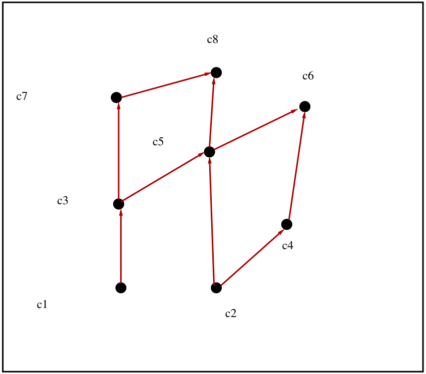
- "divides" on a set $\{1, 2, 3, 6, 9, 12, 15, 24\}$.
- x is older than y on a set of people.
- $\leq$ on the set Z^+.

Example: Course pre-requisite structure

List of courses to be completed to graduate.

$$S = \{c_1, c_2, c_3, \ldots, c_n\}.$$

$R = \{\ (c_i, c_j)\ \mid\ (c_i = c_j)$ or c_i is a pre-requisite for $c_j\ \}$



- Write down the relation R.
- Note that every (a, a) should be in R. ex: (PDS, PDS).
- What about (Disc. Maths, Adv. Algo)? , yes it belongs to R.

Comparable elements.

- a and b are said to be comparable iff $a \preceq b$ or $b \preceq a$.
- Ex: Disc. Maths $\preceq$ RP.
- Non-Ex: Prob. Th. $\npreceq$ PDS.

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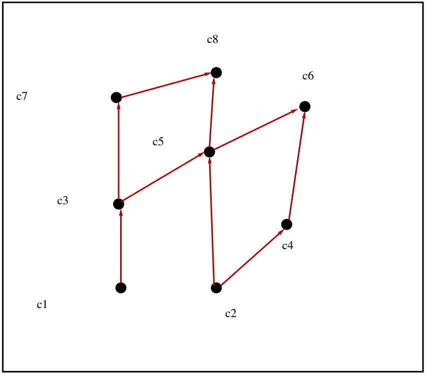


Minimal Elements

- An element "a" such that for no $b \in S$, $b \prec a$.
  Disc. Maths, Prob. Th.
- Course that does not have a pre-req.

Maximal Elements

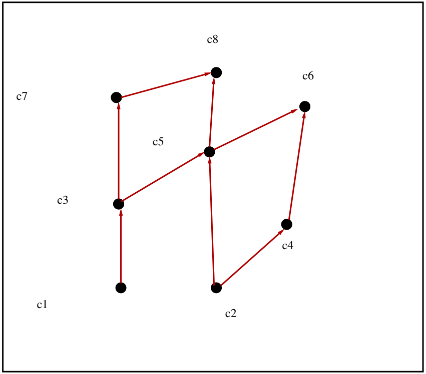
- An element "a" such that for no $b \in S$, $a \prec b$.
  Adv. Algo, R.P.
- Course that is not a pre-req. for any course.

Example: Course pre-requisite structure

List of courses to be completed to graduate.

$$S = \{c_1, c_2, c_3, \ldots, c_n\}.$$

$R = \{ \ (c_i, c_j) \ \mid \ (c_i = c_j)$ or c_i is a pre-requisite for $c_j \ \}$



Least Element

- An element "a" such that for all $b \in S$, $a \preceq b$.
- Least element is unique if it exists.

Greatest Elements

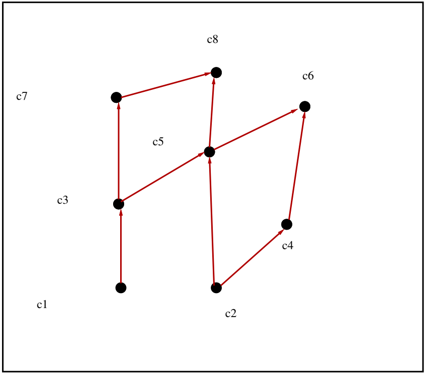
- An element "a" such that for all $b \in S$, $b \preceq a$.
- Greatest element is unique if it exists.

Example: Course pre-requisite structure

List of courses to be completed to graduate.

$$S = \{c_1, c_2, c_3, \ldots, c_n\}.$$

$R = \{ \ (c_i, c_j) \ \mid \ (c_i = c_j)$ or c_i is a pre-requisite for $c_j \ \}$



Hasse Diagram for a poset

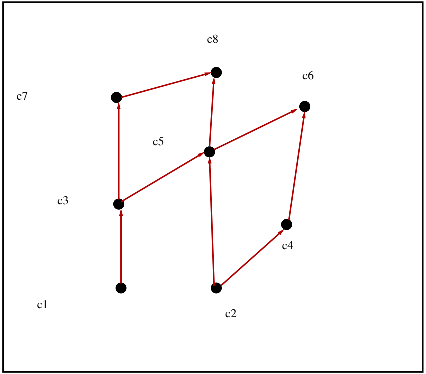
- A node for every element.
- An edge from c_i to c_j if $(c_i, c_j) \in R$.
- Omit reflexive edges.
- Omit transitive edges.
- Finally, remove the arrows (all edges go "upwards").

Example: Course pre-requisite structure

List of courses to be completed to graduate.

$$S = \{c_1, c_2, c_3, \ldots, c_n\}.$$

$R = \{ \ (c_i, c_j) \ \mid \ (c_i = c_j) \text{ or } c_i \text{ is a pre-requisite for } c_j \ \}$



Chain

- A subset of S such that every pair in this subset is comparable.
- { Disc. Maths, PDS, Algo, R.P.}
  {Disc. Maths, Adv. DS }
- Not a chain:
  { Disc. Maths, Algo, Adv. DS}

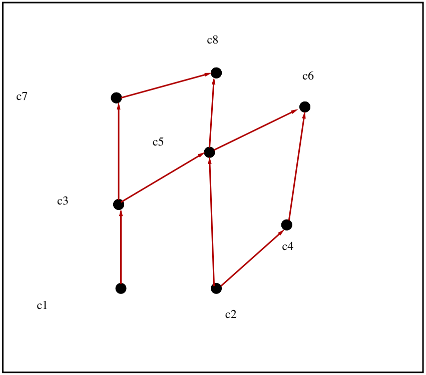
Qn: What does the length of the longest chain signify?

Example: Course pre-requisite structure

List of courses to be completed to graduate.

$$S = \{c_1, c_2, c_3, \ldots, c_n\}.$$

$R = \{ \ (c_i, c_j) \ \mid \ (c_i = c_j)$ or c_i is a pre-requisite for c_j $\}$



Anti-Chain

- A subset of S such that every pair in this subset is incomparable.
- { Disc. Maths, Adv. Prob.}
  {Adv. DS, Algo, Adv. Prob. }
- Neither a chain nor an anti-chain:
  { Disc. Maths, Algo, Adv. DS}

Qn: What does the length of the longest anti-chain signify?

Summary

- Equivalence Relations and Properties.
- Partial Order and Hasse Diagrams.
- Chains and Antichains.
- Partial Order useful to model various real-world examples.
- References : Section 9.5, 9.6 [KR]