Structured Sets

CS1200, CSE IIT Madras

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April 21, 2020

Structured Sets

- Relational Structures
  - Properties and closures ✓
  - Equivalence Relations ✓
  - Partially Ordered Sets (Posets) and Lattices
- Algebraic Structures
  - Groups and Rings

Partially Ordered Sets

- S_1 – all words in English dictionary.
- Relation R_1 on S_1:
  - $(w_1, w_2) \in R_1$ if $w_1 = w_2$ or w_1 appears before w_2 in dictionary.

- S_2 – all subsets of $\{a, b, c\}$.
- Relation R_2 on S_2:
  - $(X, Y) \in R_2$ if $X \subseteq Y$.

Defn: If R on set S is reflexive, and anti-symmetric, and transitive, then R is a partial ordering on set S. Set S along with R is known as a partially ordered set or poset.
$a \preceq b$ is used to denote $(a, b) \in R$ when R is reflexive, anti-symmetric and transitive.

Examples:

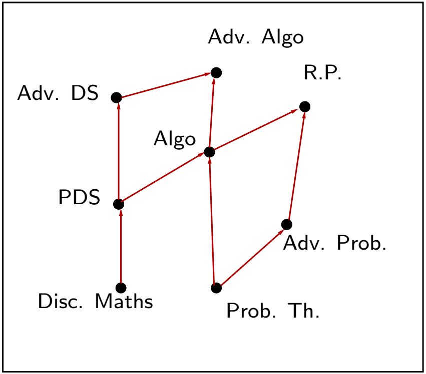
- "divides" on a set $\{1, 2, 3, 6, 9, 12, 15, 24\}$.
- x is older than y on a set of people.
- $\leq$ on the set Z^+.

Example: Course pre-requisite structure

List of courses to be completed to graduate.

$$S = \{c_1, c_2, c_3, \ldots, c_n\}.$$

$R = \{ \ (c_i, c_j) \ \mid \ (c_i = c_j)$ or c_i is a pre-requisite for $c_j \ \}$



- Comparable elements.
- Minimal elements.
- Least element (if exists).
- Chain and Anti-chain.
- Length of Longest Chain: Minimum number of semesters needed to complete the course work.
- Length of Longest Anti-chain: Maximum number of courses that one can take simultaneously (without violating pre-req).

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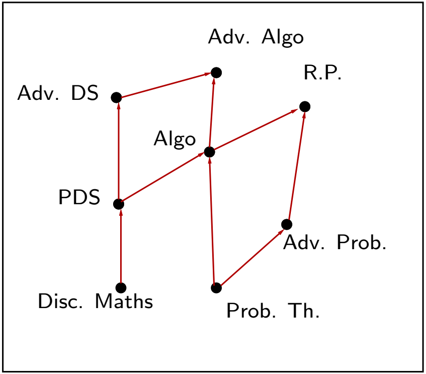
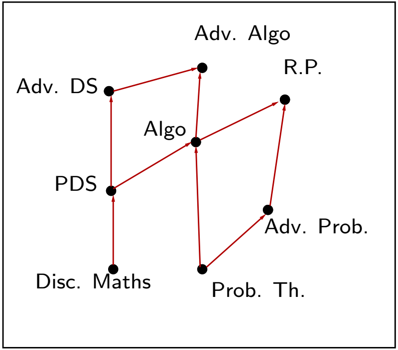


- Qn: Is there a total order on the courses "compatible" with the given partial order?
  An ordering: Disc. Maths, Prob. Th., PDS, Adv. Prob., Adv. DS, Algo, RP, Adv. Algo
- Is this order unique? No. Write down another order.

Total ordering of a partial order

For a poset $(S, \preceq)$, the relation $\preceq_t$ is said to be a total order on S if $a \preceq b$ implies $a \preceq_t b$. Note: it is not an iff statement.

A total order is also called as a linearization of the partial order.



Prob. Th. $\preceq_t$ Disc. Maths $\preceq_t$ PDS $\preceq_t$ Adv. Prob. $\preceq_t$ Adv. DS $\preceq_t$ Algo $\preceq_t$ Adv. Algo $\preceq_t$ RP ✓

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Qn: How to construct the total order?

Topological sorting of a partial order.

Claim: Every finite poset $(S, \preceq)$ has at least one minimal element.

Proof: Consider any element $a_i \in S$. If a_i is a minimal element we are done, else there exists an $a_j \in S$ such that $a_j \prec a_i$. Continue the argument with a_j. We must stop eventually since the poset is finite.

Topological Sort of a finite poset $(S, \preceq)$:

- $k = 1$
- while there are elements in S
  - Let a_i be a minimal element in S w.r.t. $\preceq$
  - $b_k = a_i$ (assign the a_i as the k-th element in the order.)
  - $S = S \setminus \{a_i\}$
- Output $b_1, b_2, \ldots, b_n$ as the total order.

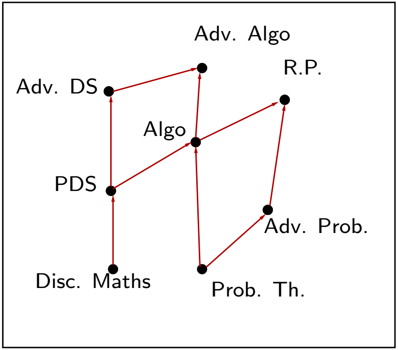
Back to Chains and Anti-chains

A poset $(S, \preceq)$

Chain: A subset $S' \subseteq S$ such that every pair of elements in S' is comparable.

Maximal chain: A chain that is not a subset of any chain of the poset.

Longest chain: A chain S' s.t. no other chain has more elements than $|S'|$.



- $S' = \{Disc.Maths, Adv.DS\}$ a valid chain, but not maximal.
- $S' = \{Prob.Th., Algo, Adv.Algo\}$ a maximal chain but not longest.
- Find the longest chain S' in the example. $|S'| = 4$.

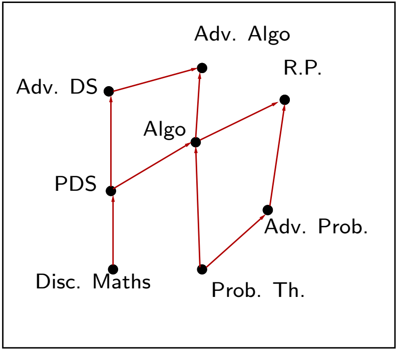
Back to Chains and Anti-chains

A poset $(S, \preceq)$

Anti-chain: A subset $S' \subseteq S$ such that every pair of elements in S' is incomparable.

Maximal anti-chain: An anti-chain that is not a subset of any anti-chain of the poset.

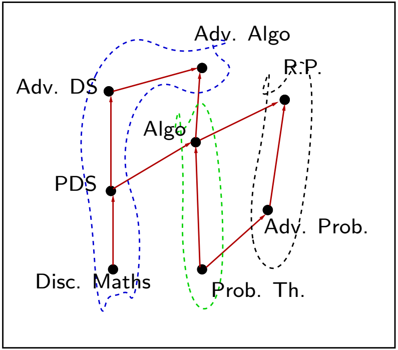
Longest anti-chain: An anti-chain S' s.t. no other anti-chain has more elements than $|S'|$.



- $S' = \{Disc.Maths, Adv.Prob\}$. Is it an anti-chain? Is it maximal? But not longest.
- $S' = \{Adv.DS, Algo, Adv.Prob.\}$ is the longest anti-chain.

A finite poset $(S, \preceq)$ and let k be the length of the longest anti-chain.

Claim: The set S can be partitioned as k chains.



- The longest anti-chain is size 3.
- The blue, green and black (three of them) partition S.
- Is it true for every poset?
  Ex: Attempt a proof.

An application

We are given a group of $mn + 1$ people. Show that there is:

- either a list of $m + 1$ people such that every person in the list (except the first one) is a descendant of the previous person in the list, or
- there is a set of $n + 1$ people such that there is no pair in this set where one person is a descendant of the other.

Proof: Let S be the set of $mn + 1$ people. Define $a \preceq b$ if either $a = b$ or a is descendant of b. Argue that $(S, \preceq)$ is a poset.

Let set $S' \subseteq S$ of people be such that for every pair a, b in S', neither is a descendant of the other. Furthermore S' be the largest such set.

If $|S'| = n + 1$, we are done. Else let $|S'| = k \leq n$. By claim earlier, S can be partitioned into $k \leq n$ chains.

By pigeonhole principle, we must have at least one chain containing $m + 1$ people. This chain is the desired list.

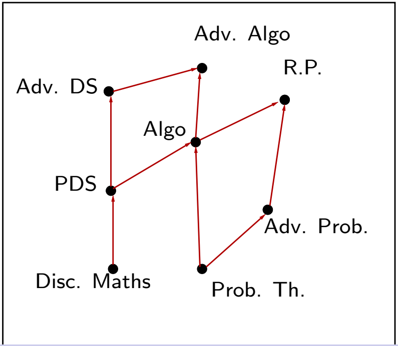
This completes the proof.

A poset $(S, \preceq)$ and let $S' \subseteq S$.

Upper bound of S' : An element $u \in S$ (if it exists) such that $a \preceq u$ for every $a \in S'$.

Least upper bound (lub) of S': An upper bound u which is less than every other upper bound of S'.

Similar definitions for lower bound and greatest lower bound (glb) of a set S'.



$S' = \{Disc.Maths, Prob.Th., Algo\}$.

- Adv. Algo is an upper bound for S' but not least upper bound.
- Algo is an lub for S'.
- The set S' has no lower bound and hence no glb.

Lattice: Poset with additional Properties

A poset $(S, \preceq)$ such that every pair of elements has a both an lub and glb is called a lattice.

Examples:
$S = \{1, 2, 3, 4, 5\}$, $(a, b) \in R$ if a divides b.

- Verify that (S, R) is a poset.
- For the pair (2, 3), we see that 1 is a glb, but the pair has no upper bounds and hence no lub.
- Hence (S, R) is a poset but not a lattice.

Let X be any set and $S = \mathcal{P}(X)$. Let $(A, B) \in R$ if $A \subseteq B$.

- Verify that (S, R) is a poset.
- For any $A, B \in \mathcal{P}(S)$, we have $A \cap B$ is glb and $A \cup B$ is lub.
- Hence (S, R) is a poset that is a lattice.

Ex: Read Example 25 in Section 9.6[KR] for application of lattices.

Summary

- Posets and properties.
- Chains and Anti-chains and useful relations between size of longest one and "covers" by the other.
- Posets with additional properties : Lattices.
- Reference: Section 9.6 [KR]