Structured Sets

CS1200, CSE IIT Madras

Meghana Nasre

April 21, 2020

Structured Sets

- Relational Structures
  - Properties and closures ✓
  - Equivalence Relations ✓
  - Partially Ordered Sets (Posets) and Lattices
- Algebraic Structures
  - Groups and Rings

Partially Ordered Sets

- S_1 – all words in English dictionary.
- Relation R_1 on S_1:
  - $(w_1, w_2) \in R_1$ if $w_1 = w_2$ or w_1 appears before w_2 in dictionary.
- S_2 – all subsets of $\{a, b, c\}$.
- Relation R_2 on S_2:
  - $(X, Y) \in R_2$ if $X \subseteq Y$.

Defn: If R on set S is reflexive, and anti-symmetric, and transitive, then R is a partial ordering on set S. Set S along with R is known as a partially ordered set or poset.
$a \preceq b$ is used to denote $(a, b) \in R$ when R is reflexive, anti-symmetric and transitive.

Examples:

- "divides" on a set $\{1, 2, 3, 6, 9, 12, 15, 24\}$.
- x is older than y on a set of people.
- $\leq$ on the set Z^+.

Example: Course pre-requisite structure

List of courses to be completed to graduate.

$$S = \{c_1, c_2, c_3, \ldots, c_n\}.$$

$R = \{ \ (c_i, c_j) \ \mid \ (c_i = c_j)$ or c_i is a pre-requisite for c_j $\}$

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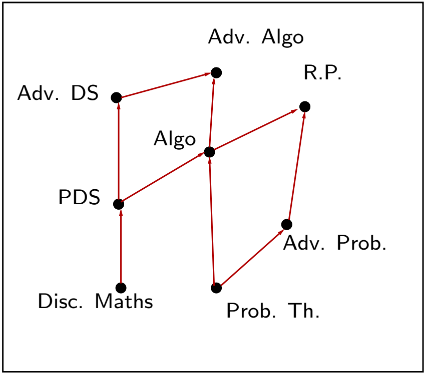

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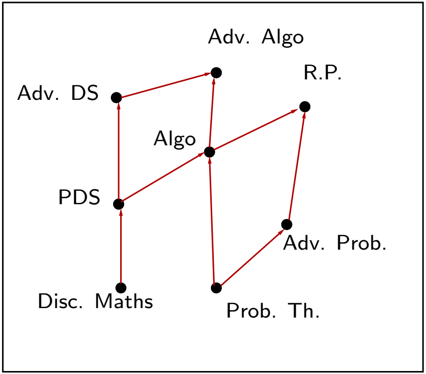

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- Minimal elements.
- Least element (if exists).
- Chain and Anti-chain.

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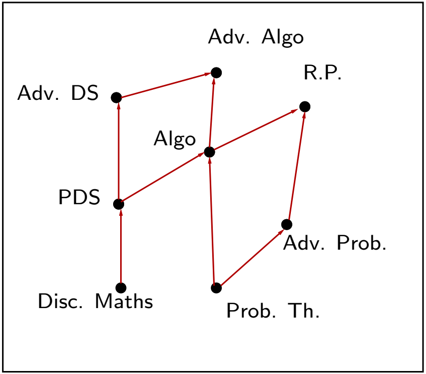

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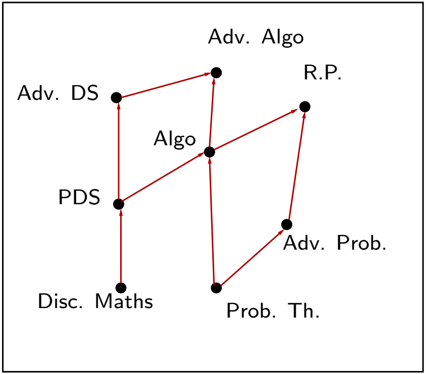

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- Minimal elements.
- Least element (if exists).
- Chain and Anti-chain.
- Length of Longest Chain: Minimum number of semesters needed to complete the course work.
- Length of Longest Anti-chain: Maximum number of courses that one can take simultaneously (without violating pre-req).

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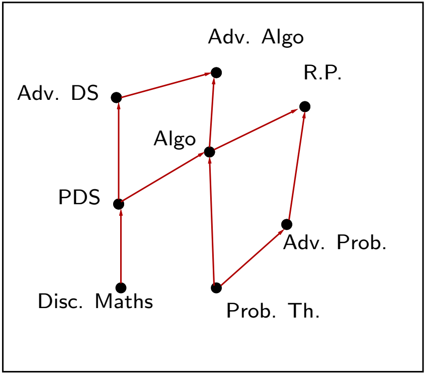

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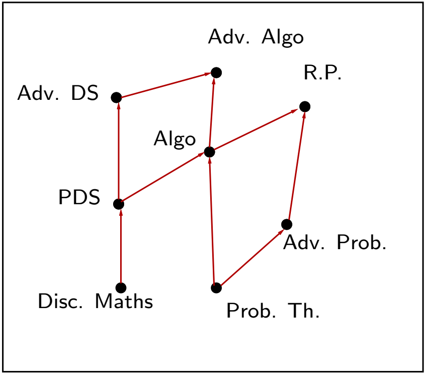

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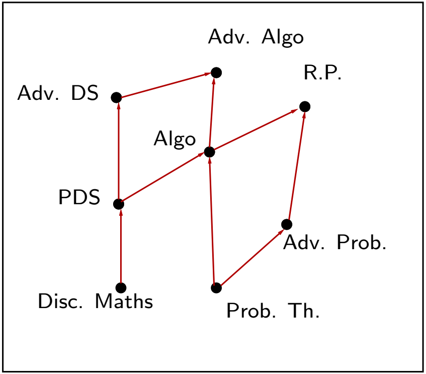

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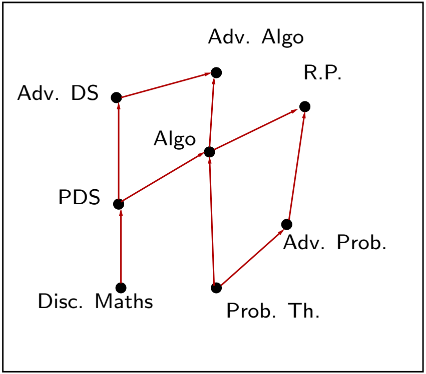

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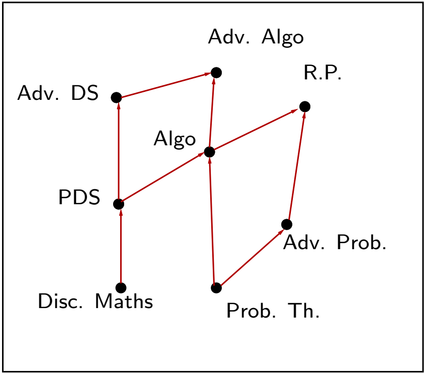

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- Is this order unique? No. Write down another order.

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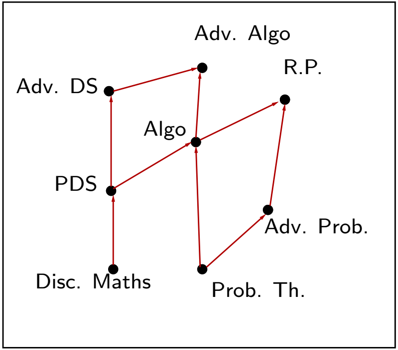

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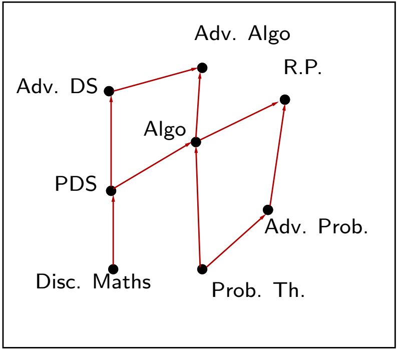

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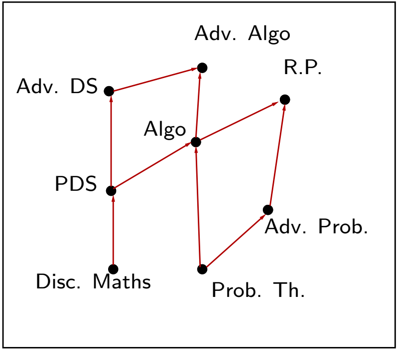

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- Output $b_1, b_2, \ldots, b_n$ as the total order.

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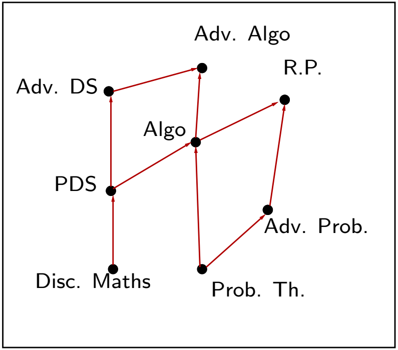

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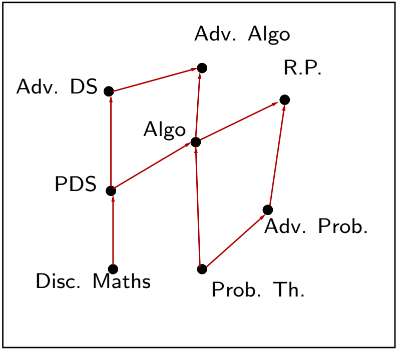

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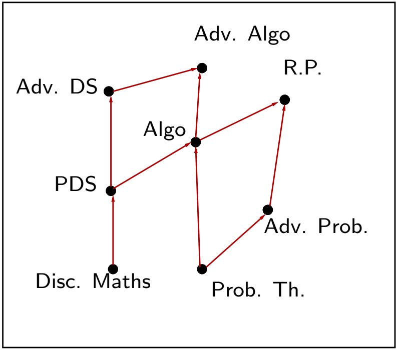

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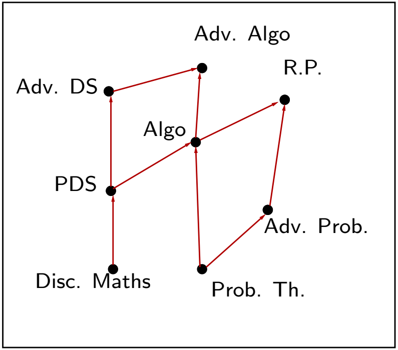

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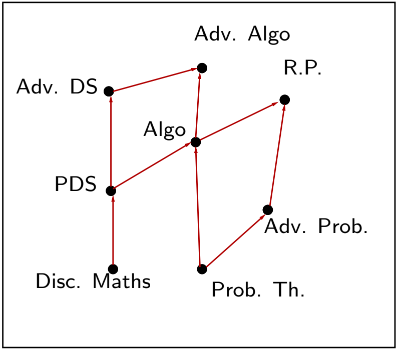

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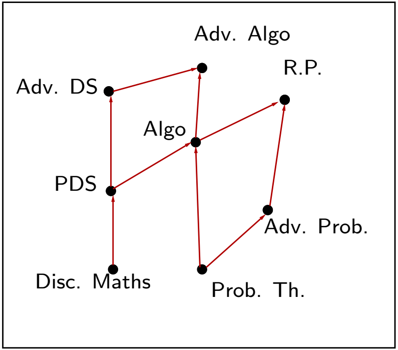

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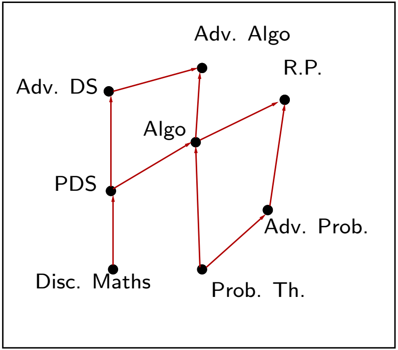

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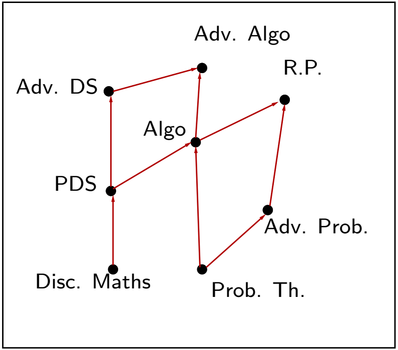

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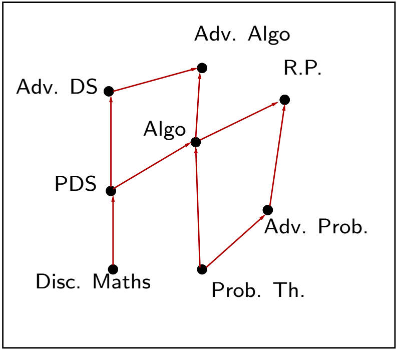

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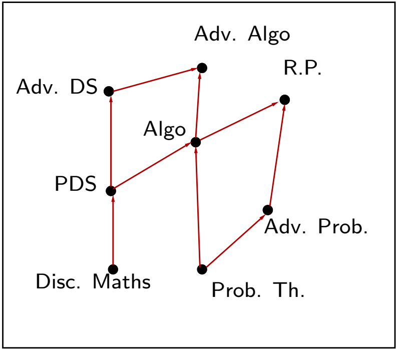

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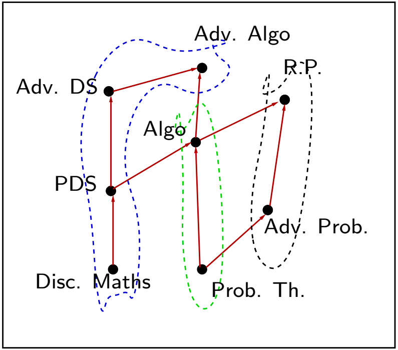

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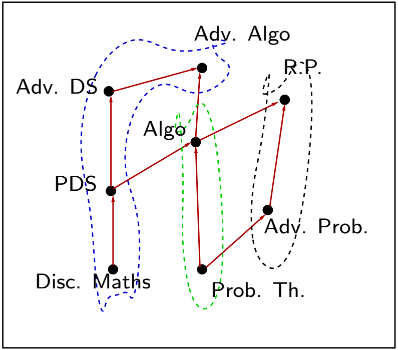

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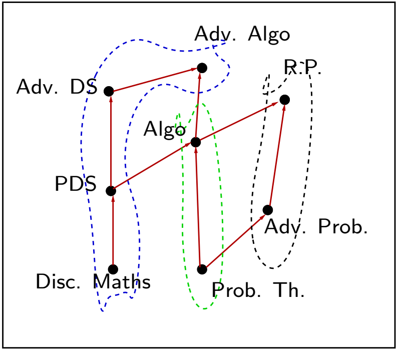

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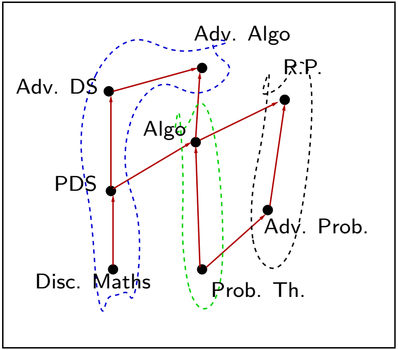

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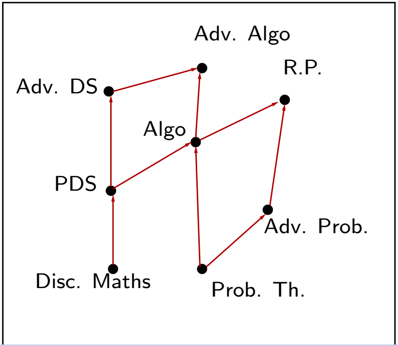

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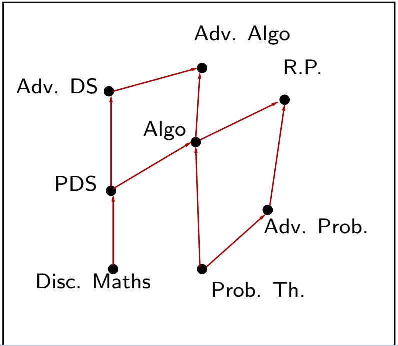

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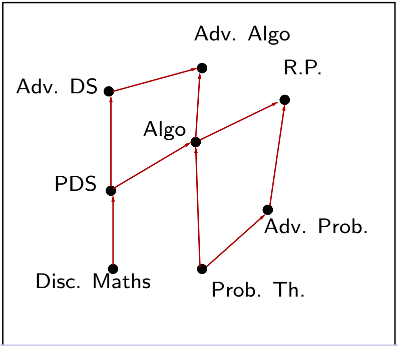

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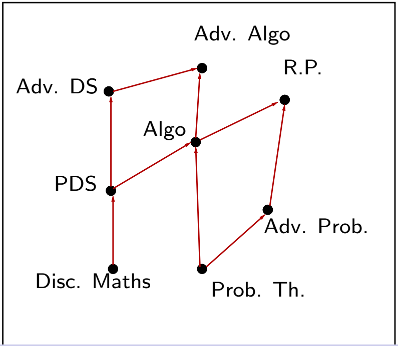

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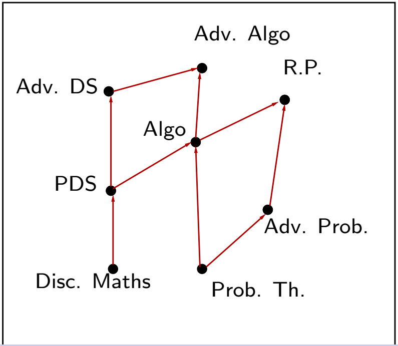

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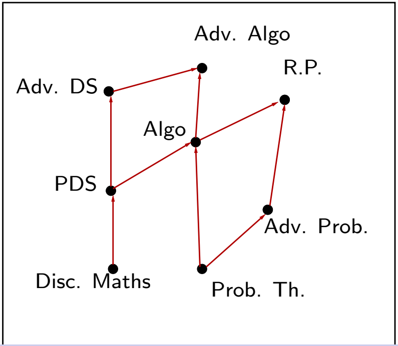

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Ex: Read Example 25 in Section 9.6[KR] for application of lattices.

Summary

- Posets and properties.
- Chains and Anti-chains and useful relations between size of longest one and “covers” by the other.
- Posets with additional properties : Lattices.
- Reference: Section 9.6 [KR]