Structured Sets

CS1200, CSE IIT Madras

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Structured Sets

- Relational Structures
  - Properties and closures ✓
  - Equivalence Relations ✓
  - Partially Ordered Sets (Posets) and Lattices ✓
- Algebraic Structures
  - Groups and Rings

Algebraic Structures: Recap

Set A with a binary operator $*$

- If $*$ is closed and associative, and an identity element e exists, and every element $b \in A$ has an inverse then $(A, *)$ is a group.

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- Lagrange's Theorem: The order of any subgroup of a finite group divides the order of the group.

Cosets of a subset

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Similarly argue that $H_d \subseteq H_c$. This completes the argument that if there is even one common element then the sets are equal.

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  - Verify that $(Z_n, \odot_n)$ is a semigroup
  - Verify that $\odot_n$ distributes over $\oplus_n$

Let $(A, +, \cdot)$ be an algebraic structure. It is called a ring if

- $(A, +)$ is an Abelian group. recall Abelian says $+$ is commutative.
- $(A, \cdot)$ is a semigroup.
- The operation $\cdot$ is distributive over the operation $+$.

Additionally, if $(A, \cdot)$ is a monoid, then $(A, +, \cdot)$ is a called a ring with identity.

Examples:

- $(Z, +, \cdot)$ is a ring with identity.
- Recall the set Z_n for any positive integer n. We have seen the operation $\oplus_n$ and verified that $(Z_n, \oplus_n)$ is a group. Now define $\odot$ as

$$a \odot_n b = ab \mod n$$

  - Verify that $(Z_n, \odot_n)$ is a semigroup
  - Verify that $\odot_n$ distributes over $\oplus_n$

  Thus, $(Z_n, \oplus_n, \odot_n)$ is a ring.

Summary

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- Reference: Section 11.3, Elements of Discrete Mathematics by C. L. Liu.