# Advanced Counting Techniques

CS1200, CSE IIT Madras

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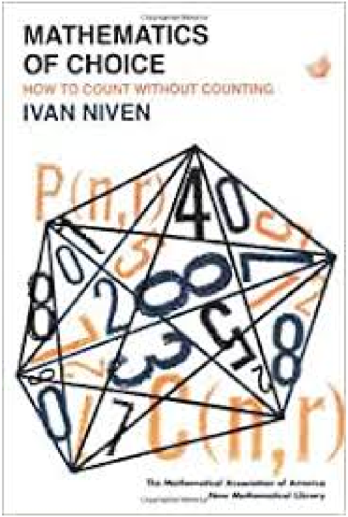


- Principle of Inclusion-Exclusion
- Recurrences and its applications
- Solving Recurrences

# Example: Number of solutions to an equation

$$x_1 + x_2 + x_3 = 11$$

- How many integral solutions if each $x_i \geq 0$? ✓
- How many integral solutions if each $x_i \geq 1$? (replace $x_i - 1 = y_i$ and equation is $y_1 + y_2 + y_3 = 8$) ✓
- What if we want to count the number of solutions for $x_1 + x_2 + x_3 \leq 11$? where each $x_i \geq 0$?
  (add another variable $x_4$ and set it to an equality. $x_4 \geq 0$.) ✓
- How many integral solutions if each $x_i \geq 0$ and $x_1 \geq 6$? (we have seen an example of this earlier) ✓

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All the above can be solved using combinations with repetitions.

- How many integral solutions if each $x_i \geq 0$ and $x_1 \leq 3$, $x_2 \leq 4$ and $x_3 \leq 6$?

Verify that this cannot be solved using the same. We will use the principle of inclusion exclusion.

# Principle of Inclusion Exclusion

Principle of Inclusion Exclusion gives a formula to find the size of union of **finite** sets.

It is a generalization of the familiar formula below.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Ex: Write down the formula for $|A \cup B \cup C|$.

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Principle of Inclusion Exclusion: Let $A_1, A_2, \ldots, A_n$ be $n$ **finite** sets. Then,

$$\begin{aligned} |A_1 \cup A_2 \cup \ldots \cup A_n| = & \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ + & \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| + \ldots \\ + & (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n| \end{aligned}$$

## Correctness of Principle of Inclusion Exclusion

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We prove that the formula is correct, or any element $x \in A_1 \cup A_2 \cup \ldots \cup A_n$ is counted exactly once. Let $x$ belong to $r$ amongst the $n$ sets.

- It is counted $r = \binom{r}{1}$ times by $\sum |A_i|$.
- It is counted $\binom{r}{2}$ times by $\sum |A_i \cap A_j|$ for the pairs $A_i$ and $A_j$ both containing $x$. Note the negative sign for this count.
- In general, it is counted $\binom{r}{m}$ times for the intersection of $m$ of the $r$ sets containing $x$.
- Thus, $x$ gets counted exactly:

$$\binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \ldots + (-1)^{r+1} \binom{r}{r} \text{ times}$$

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We prove that the formula is correct, or any element $x \in A_1 \cup A_2 \cup \ldots \cup A_n$ is counted exactly once. Let $x$ belong to $r$ amongst the $n$ sets.

- Thus, $x$ gets counted exactly:

$$\binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \ldots + (-1)^{r+1} \binom{r}{r} = k \text{ times}$$

However, note that

$$\binom{r}{0} - k = 0 \qquad \text{why?}$$

Thus, $k = \binom{r}{0} = 1$, that is $x$ gets counted exactly once on the RHS.

# An alternate form

Many times we have several properties say $P_1, P_2, \ldots, P_n$.

- Let $A_i$ be the set of elements that satisfy $P_i$.
- We also know the total number of elements say $N$ (without any regard to whether they satisfy any of the properties)
- And we are interested in counting the number of elements that do not satisfy any of the properties. Denote the number by $N(\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n)$.

Thus,

$$N(\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n) = N - |A_1 \cup A_2 \cup \ldots \cup A_n|$$

We see examples of this form next.

# Example: Arranging six people for a photo

Qn: Six people which has three couples stand in line for a photo. How many ways can they be arranged such that no wife stands next to her husband?

- circle denotes a woman, square denotes a man
- same color denotes a couple
- above arrangement is not allowed since the blue couple stands together.

Can we cast this as properties that we wish to avoid?

$P_i$: property that $i$-th couple stands next to each other for $i = 1, 2, 3$.

$A_i$: set of arrangements satisfying $P_i$.

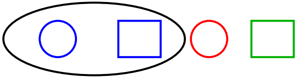
$N$: Total number of arrangements $= 6!$

Goal: Compute $N - |A_1 \cup A_2 \cup A_3|$.

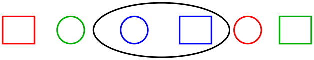
# Example: Arranging six people for a photo

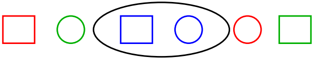
Qn: Six people which has three couples stand in line for a photo. How many ways can they be arranged such that no wife stands next to her husband?

$A_1$ is the set of arrangements satisfying $P_1$, say blue couple stand together.
Compute $|A_1|$.
We think of "gluing" the couple together.



Now we have only 5 "objects" to permute. This can be done in 5! ways.
However, for each such permutation, we can have two ways of arranging the couple.





$$|A_1| = 2 \cdot 5!$$

This holds for each $i = 1, 2, 3$.

# Example: Arranging six people for a photo

Qn: Six people which has three couples stand in line for a photo. How many ways can they be arranged such that no wife stands next to her husband?

- Let $A_1 \cap A_2$ denote the arrangements in which first and second couple stand next to each other.
  $|A_1 \cap A_2| = 2 \cdot 2 \cdot 4!$ why?
  This holds for each every $A_i, A_j$ pair.
- Finally let $A_1 \cap A_2 \cap A_3$ represent the arrangements where all there couples stand next to each other. Thus, $|A_1 \cap A_2 \cap A_3| = 2 \cdot 2 \cdot 2 \cdot 3!$.

Thus $|A_1 \cup A_2 \cup A_3| = k = 3(2 \cdot 5!) - 3(2^2 \cdot 4!) + 2^3 \cdot 3!$ Finally, number of ways to arrange such that no couple stands next to each other is $6! - k$.
Ex: Calculate the same when there are 5 couples instead of 3.

## Example: Counting number of solutions to an equation

$$x_1 + x_2 + x_3 = 11$$

How many integral solutions if each $x_i \geq 0$ and $x_1 \leq 3$, $x_2 \leq 4$ and $x_3 \leq 6$?

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Can we cast this as properties that we wish to avoid?
What is our universe (that is $N$)?

$N$ = number of integral solutions where $x_i \geq 0$. $N = \binom{3-1+11}{11}$

- Let $P_1$ denote the solutions with $x_1 \geq 4$ and $A_1$ denote such solutions
- Let $P_2$ denote the solutions with $x_2 \geq 5$ and $A_2$ denote such solutions
- Let $P_3$ denote the solutions with $x_3 \geq 7$ and $A_3$ denote such solutions

The our solution $k = N - |A_1 \cup A_2 \cup A_3|$.

- $N(P_1) = |A_1| = \binom{3-1+7}{7}$
- $N(P_1P_2) = |A_1 \cap A_2|$ = number of solutions with $x_1 \geq 4$ and $x_2 \geq 5$
  $N(P_1P_2) = \binom{3-1+2}{2}$ verify this!
- $N(P_1P_2P_3)$ is the number of solutions with $x_1 \geq 4$ and $x_2 \geq 5$ and $x_3 \geq 6$.
  Clearly $N(P_1P_2P_3) = 0$.
- Compute $|A_1 \cup A_2 \cup A_3| = \sum_{i=1}^{3} N(P_i) - \sum_{1 \leq i < j \leq 3} N(P_iP_j) + N(P_1P_2P_3)$

# Summary

- Principle of inclusion exclusion with applications.
- Allows us to count elements avoiding certain properties.
- Need to come up with appropriate properties (specific to the example)
- Once properties are identified (correctly) use known techniques to count sets satisfying properties.
  - In arranging couples, we used product rule.
  - In number of solutions to equation, we used combinations with repetition.
- References Section 8.6[KR]