Advanced Counting Techniques

CS1200, CSE IIT Madras

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April 7, 2020

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- Principle of Inclusion-Exclusion ✓
- Recurrences and its applications
- Solving Recurrences

Recurrences

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Today's class: Modeling a variety of problems having the same solution.

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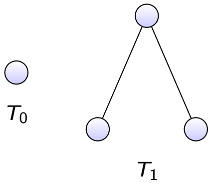

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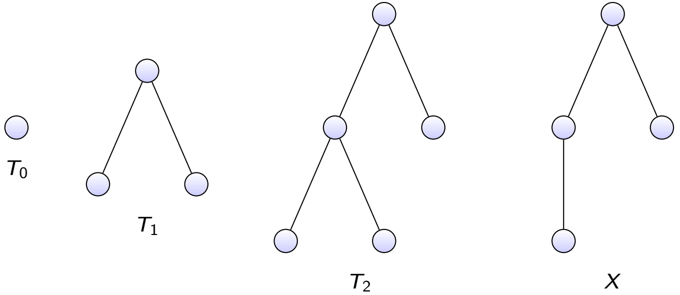

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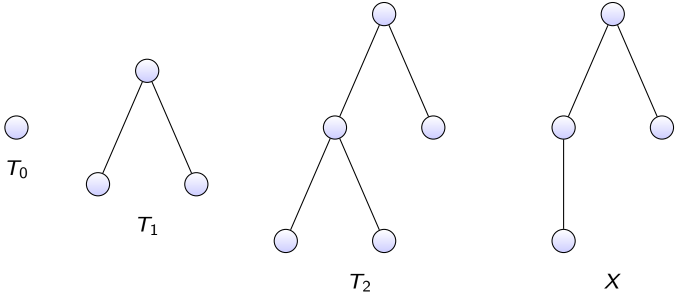

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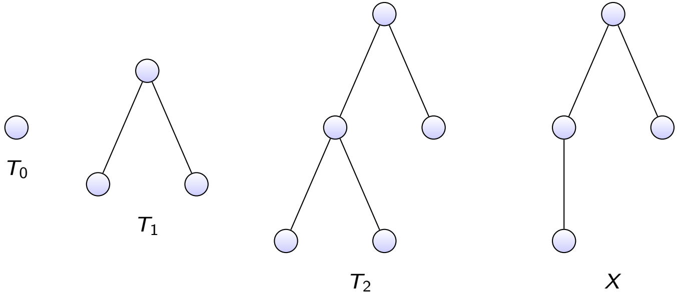

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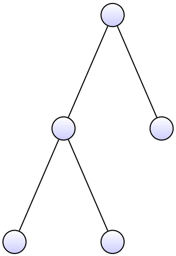
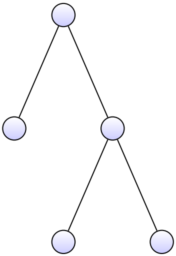


T_2^1



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- 2 internal nodes in left subtree and $n-3$ internal nodes in right subtree. This gives $f(2) \cdot f(n-3)$ ways of constructing trees with n internal nodes.

$\vdots$

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- Why is adding the terms valid?
- What is a closed form expression for this recurrence?

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$g(n)$ denotes the number of ways to parenthesize $n + 1$ integers and not n integers!

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Base cases: $g(0) = 1$, $g(1) = 1$.

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- On the left side of the last "$\cdot$" we have $k+1$ integers $x_0, \ldots x_k$. These can be parenthesized in $g(k)$ ways.
- On the right side of the last "$\cdot$" we have $n-k$ integers $x_{k+1}, \ldots, x_n$. These can be parenthesized in $g(n-k-1)$ ways. Note that it is not $g(n-k)$ ways.

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Since k can take values between 0 and $n-1$, we have

$$g(n) = g(0) \cdot g(n-1) + g(1) \cdot g(n-2) + \ldots + g(n-2) \cdot g(1) + g(n-1) \cdot g(0)$$

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Compare it with the recurrence for $f(n)$ earlier.

Other Examples

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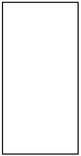

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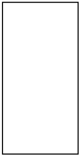

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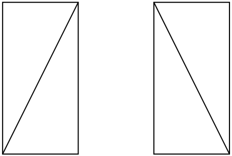

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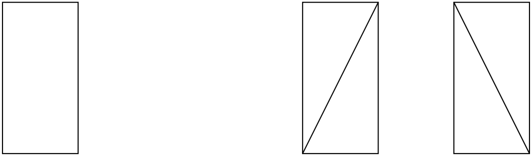

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Ex: Work out the recursive formulation for both the examples.

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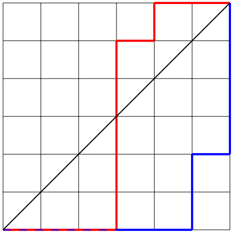

- Two paths – one red and another blue.
- Red path is invalid. (RRRUUUUURURR)
- Blue path is valid. (RRRRRUURUUUU)

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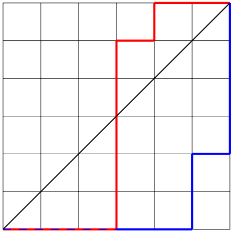

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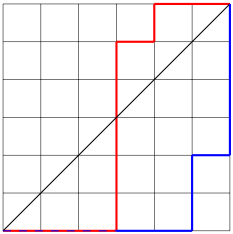
- Red path is invalid. (RRRUUUUURURR)
- Blue path is valid. (RRRRRUURUUUU)

Goal: Compute total number of valid paths, that is, diagonal avoiding paths.

Example 3: Diagonal Avoiding Grid Paths

Goal: Input is a $n \times n$ grid. To compute number of paths from $(0, 0)$ to (n, n).
Constraints:

- Use steps of one unit and go right or up at each step.
- Each path contains n right steps (R) and n up (U) steps.
  we have already solved this when there were no more constraints
- Additional condition: Paths should not cross the main diagonal.



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A first guess it to count all paths from $(0, 0)$ to (n, n) which is $\binom{2n}{n}$ and then claim that half of the total paths are diagonal avoiding.

Example 3: Diagonal Avoiding Grid Paths

Input: An $n \times n$ grid.

Goal: To compute number of diagonal avoiding paths from $(0,0)$ to (n,n), denote it by $h(n)$.

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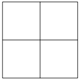

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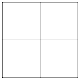

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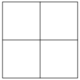

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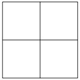

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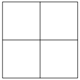

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In contrast every valid path satisfies the property that every prefix has at least as many Rs as the number of Us. Thus, every diagonal avoiding grid path is in one-to-one correspondence with a string of balanced parenthesis of length $2n$.

This is simply obtained by replacing every "(" by R and every ")" by U.

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- It has a prefix in which there are more Us than Rs. (that is why it crosses the diagonal!)
- Select the smallest such prefix. (RRRUUUU – in the above).

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- Keep the prefix of the path as it is and in the remaining path flip the Us and Rs. Call this the modified path.

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invalid path (RRRUUUUURURR) → modified path RRRUUUUURURUU

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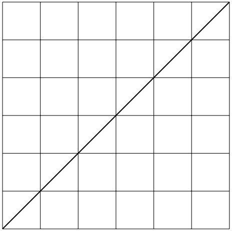

- Note that the modified blue path follows the red path for the prefix part, and then flips it.

How does it help?

- Every modified path always ends in $(n-1, n+1)$.
  revisit the previous slide and construct a proof!
- One to one correspondence between invalid paths and paths (no more constraints of diagonal avoiding) from $(0,0)$ to $(n-1, n+1)$.
  establish this correspondence!

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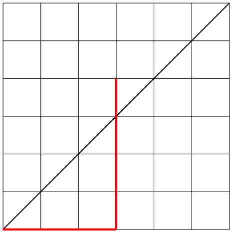

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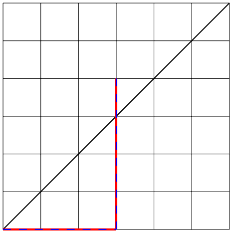

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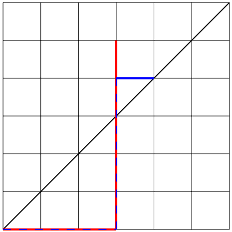

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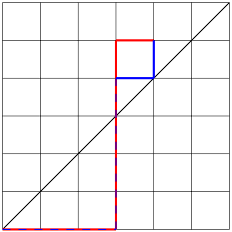

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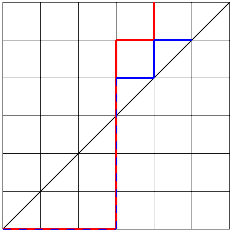

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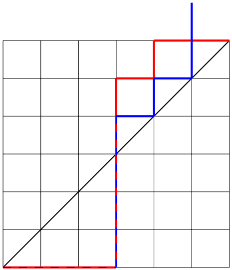

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Number of valid paths $= h(n) =$

$$\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

Catalan Number

The number $\frac{1}{n+1}\binom{2n}{n}$ is called the n-th Catalan number.

- Gives a closed form solution to several examples (seen today).
- Many more applications.
- References Section 8.1[KR].