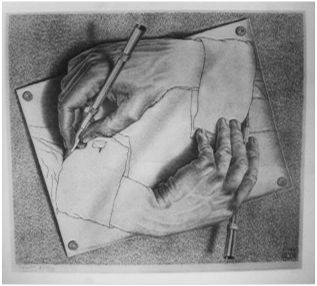
Recursion and Proofs by Induction – Part II

CS1200, CSE IIT Madras

Meghana Nasre

March 20, 2020

Recursion Continued



Drawing Hands by M. C. Escher

- Familiar recursive functions ✓
- Some important recursive functions ✓
- Proving closed form solutions using induction ✓
- Defining objects and sequences using recursion

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Proof: Let A be the set of all non-negative powers of 3. Show that $S = A$.
Note that $A = \{\ 3^n \ \mid \ n \in \mathbb{Z}_{\geq 0}\}$

- Show that $A \subseteq S$
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Thus, we know that all non-negative powers of 3 belong to S. That is, $A \subseteq S$.

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Thus, S contains only those integers that are non-negative powers of 3, i.e., $S \subseteq A$.

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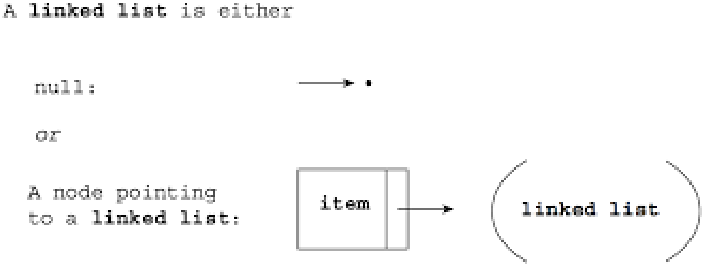

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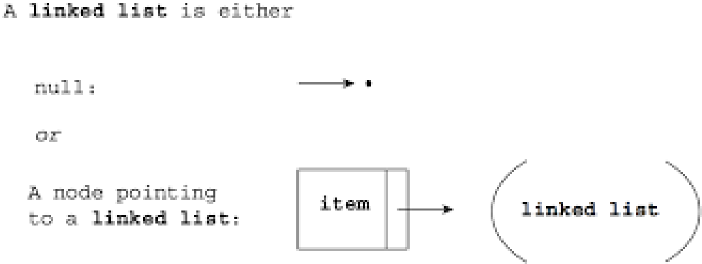
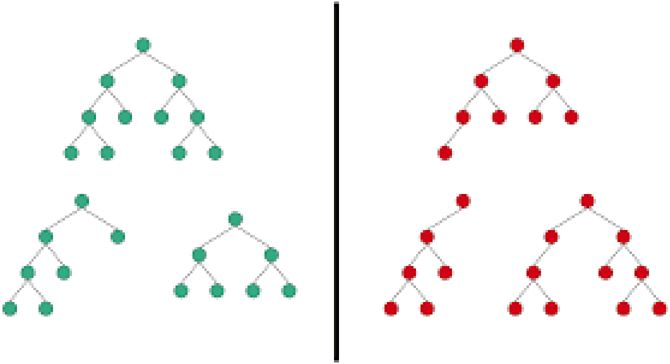


Ex: Define length of a linked list recursively.

Trees defined recursively



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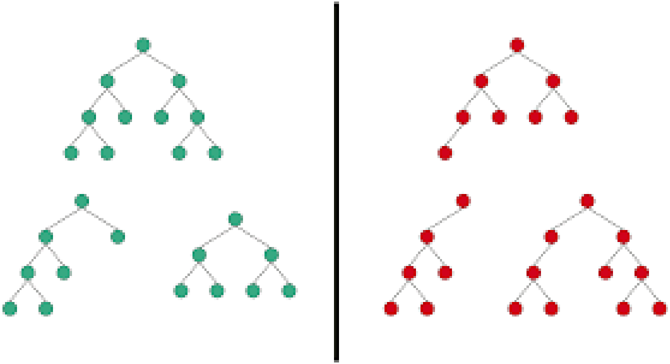

Trees are drawn upside down in CS!

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$$\begin{aligned} n(T) &= 1 + n(T_1) + n(T_2) && \text{this answers last Ex: on prev. slide partly} \\ &\leq 1 + 2^{h(T_1)} + 2^{h(T_2)} && \text{by inductive hypothesis} \end{aligned}$$

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Complete the proof of the correct claim – see Theorem 2, Section 5.3 [KR]

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- The answer is $c_1 = 3$ and $f(n) = n$. Now prove that these values are indeed correct by using induction on n.

Summary

- Recursive Sets and proofs using induction and structure of the set.
- Recursively defined objects, specifically trees and their properties.
- Recursive sequences.
- left as reading exercise: Recursion and strings.
- Reference: Section 5.3 [KT].