Counting

CS1200, CSE IIT Madras

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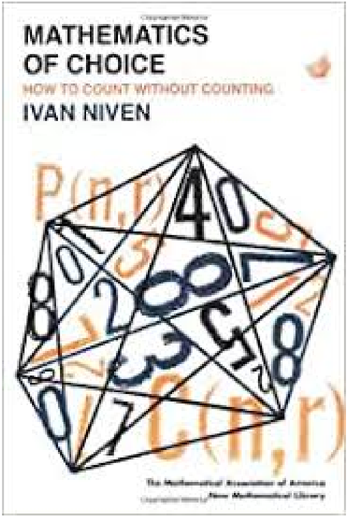
Counting (without counting)



- Basic Counting Techniques ✓
- Pigeon Hole Principle (revisited)
- Permutations and Combinations
- Combinatorial Identities

Pigeonhole principle

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Sol: Note that if we are lucky we can get a pair by pulling two socks. However, in the worst case, we may pull out 4 socks and all of them may be of different colors. Thus if we pull out 5 or more socks, we will always be guaranteed a pair of some color.

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Sol: Note that every integer from $1, 2, \ldots, 8$ has a unique "partner" which ensures that their sum is 9. Lets make four holes, one per pair. Now since we select 5 integers there is at least one hole in which both the elements of the pair are selected. This completes the proof.

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Ex: In our example selection, we saw two pairs. Can we strengthen our claim that there are always two pairs that sum upto 9? If yes, modify the proof. If no, construct a counter example.

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Sol: There are 100 total addresses, lets have 50 holes so that each hole corresponds to two consecutive addresses. That is, 1000, 1001 are one hole and so on.

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Sol: There are 100 total addresses, lets have 50 holes so that each hole corresponds to two consecutive addresses. That is, 1000, 1001 are one hole and so on. Now since there are 51 houses, there is at least one hole containing two houses, implying that there are at least two houses with consecutive integers as their numbers. This completes the proof.

A clever application of the principle

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Qn: Identify a decreasing subseq. in the given sequence. Does every sequence have an increasing as well as decreasing subsequence?

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- Note that if you have a monotonically increasing sequence as your input, you cannot find even a 2 length decreasing subsequence (forget large!).

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Side Question: How do we compute i_k and d_k algorithmically? It is not needed for this proof, but interesting on its own.

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- Write down i_2, d_2.

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Theorem: Given any $n^2 + 1$ length sequence of distinct integers there is either an $n + 1$ strictly increasing subsequence or an $n + 1$ strictly decreasing subsequence. We will associate two integers with every element in the sequence. Those are i_k and d_k.

- i_k denotes the longest increasing subsequence starting at a_k.
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However, we have $n^2 + 1$ elements in the input sequence. Thus there must be two elements a_s and a_t such that $i_s = i_t$ and $d_s = d_t$. Further note that we can assume a_s appears before a_t in the sequence. That is, $s < t$.

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We have established that there must be two elements a_s and a_t such that $i_s = i_t$ and $d_s = d_t$. Further note that we can assume a_s appears before a_t in the sequence. That is, $s < t$.

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Now, recall elements are distinct hence either $a_s < a_t$ or $a_s > a_t$.

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Now, recall elements are distinct hence either $a_s < a_t$ or $a_s > a_t$.

- If $a_s < a_t$, then we can take the increasing sequence starting at a_t and append a_s to that to improve $i_s = i_t + 1$.

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- If $a_s < a_t$, then we can take the increasing sequence starting at a_t and append a_s to that to improve $i_s = i_t + 1$.
- On the other hand, if $a_s > a_t$ we can improve the length of the decreasing sequence at a_s by prepending a_s to the d_t length decreasing sequence.

Example 3: Increasing / decreasing subsequences

Theorem: Given any n^2+1 length sequence of distinct integers there is either an $n+1$ strictly increasing subsequence or an $n+1$ strictly decreasing subsequence.

We will associate two integers i_k and d_k with every element in the sequence.

- i_k denotes the longest increasing subsequence starting at a_k.
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We have established that there must be two elements a_s and a_t such that $i_s = i_t$ and $d_s = d_t$. Further note that we can assume a_s appears before a_t in the sequence. That is, $s < t$.

Now, recall elements are distinct hence either $a_s < a_t$ or $a_s > a_t$.

- If $a_s < a_t$, then we can take the increasing sequence starting at a_t and append a_s to that to improve $i_s = i_t + 1$.
- On the other hand, if $a_s > a_t$ we can improve the length of the decreasing sequence at a_s by prepending a_s to the d_t length decreasing sequence.

Thus, we cannot have a pair a_s and a_t with $i_s = i_t$ and $d_s = d_t$.

Example 3: Increasing / decreasing subsequences

Theorem: Given any $n^2 + 1$ length sequence of distinct integers there is either an $n + 1$ strictly increasing subsequence or an $n + 1$ strictly decreasing subsequence.

We will associate two integers i_k and d_k with every element in the sequence.

- i_k denotes the longest increasing subsequence starting at a_k.
- d_k denotes the longest decreasing subsequence starting at a_k.

We have established that there must be two elements a_s and a_t such that $i_s = i_t$ and $d_s = d_t$. Further note that we can assume a_s appears before a_t in the sequence. That is, $s < t$.
Now, recall elements are distinct hence either $a_s < a_t$ or $a_s > a_t$.

- If $a_s < a_t$, then we can take the increasing sequence starting at a_t and append a_s to that to improve $i_s = i_t + 1$.
- On the other hand, if $a_s > a_t$ we can improve the length of the decreasing sequence at a_s by prepending a_s to the d_t length decreasing sequence.

Thus, we cannot have a pair a_s and a_t with $i_s = i_t$ and $d_s = d_t$.

Note that this was true because all of i_k and d_k were at most n. However, there must be at least one which is $\geq n + 1$.

Example 3: Increasing / decreasing subsequences

Theorem: Given any $n^2 + 1$ length sequence of distinct integers there is either an $n + 1$ strictly increasing subsequence or an $n + 1$ strictly decreasing subsequence.

Remarks

- This is a non-trivial proof using pigeon hole principle.

Example 3: Increasing / decreasing subsequences

Theorem: Given any $n^2 + 1$ length sequence of distinct integers there is either an $n + 1$ strictly increasing subsequence or an $n + 1$ strictly decreasing subsequence.

Remarks

- This is a non-trivial proof using pigeon hole principle.
- Revisit the proof and make sure you understand it completely.

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Remarks

- This is a non-trivial proof using pigeon hole principle.
- Revisit the proof and make sure you understand it completely.
- Read Examples 11, 12, 13 of 6.2 [KR]

Summary

- Pigeonhole principle and its generalization.
- Simple and interesting applications.
- A non-trivial and elegant application to increasing / decreasing subsequences.
- Reference: Section 6.2 [KR].