Counting

CS1200, CSE IIT Madras

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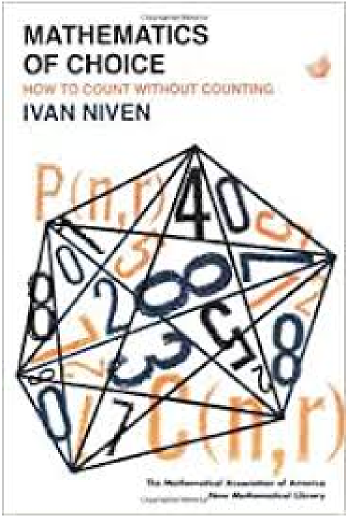
Counting (without counting)



- Basic Counting Techniques ✓
- Pigeon Hole Principle (revisited) ✓
- Permutations and Combinations ✓
- Combinatorial Identities

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- Double Counting proof: Use counting arguments to show that both sides of the identity count the same objects but in two different ways.
- Bijective proof: Establish a bijection (one-to-one onto map) between the objects counted by two sides of the identity.

A double counting proof

Claim: For any positive integer n and $0 \leq r \leq n$, we have

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- Since $\bar{A}$ contains $n - r$ elements (as A contains r elements), there are exactly $\binom{n}{n-r}$ subsets of size r.

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This completes the proof.

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Note that this is a combinatorial proof.

Some corollaries

Ex: Prove the following:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

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For each of them give a proof via

- Algebraic manipulation
- A combinatorial argument.

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Two important identities

Pascal's Identity

For two positive integers n and k, where $n \geq k$,

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- The number of subsets that contain x is $\binom{n}{k-1}$. Note that x is selected and therefore we have only n elements to select the $k-1$ elements from.
- The number of subsets that do not contain x is $\binom{n}{k}$, since we have n elements left (excluding x) to choose from and all of k elements to select.

Vandermonde's Identity

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- Ex: Interpret the RHS appropriately.

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- What are the properties of any valid path?

Any path must contain exactly m Rs and n Us.

Now write down your answer for the number of distinct paths.

Summary

- A new technique of proving identities.
- Gives insight rather than only algebriac manipulations.
- Important Identities like the Pascal's Identity and Vandermonde's Identity.
- References: Section 6.4[KR]