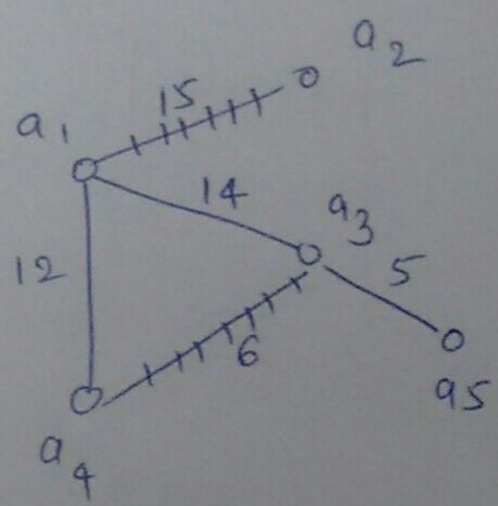
17-9-19

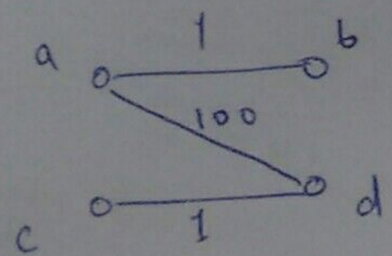
WEIGHTED MATCHING



Given a general graph G & weight / cost function on edges -

(a)- Max weight / min cost matching

(b)- Among maximum matchings, the one with max wt / min. cost.



(a) : (a - d) : Max weight

(b) : (a - b) , (c-d) : Max wt among maximum ones

BIPARTITE [COMPLETE] GRAPH :

MAX WEIGHT PERFECT MATCHING

$V = X \cup Y$

Visualize as a $n \times n$ matrix. , $w(e) \geq 0$

[Assignment Problem]

	b_1	b_2	b_3
a_1	7	4	3
a_2	15	12	9
a_3	6	5	8

Goal:

Among all perfect matchings, find max. weight matching.

LP

Primal

$$\max \sum_{e \in E} w_e x_e$$

s.t.

$$\forall v \in V, \sum_{e \in \delta(v)} x_e = 1$$

$$x_e \in \{0, 1\}$$

$\downarrow$ relaxed

$$0 \leq x_e \leq 1$$

Dual

$$\min \sum_{v \in V} y_v \approx$$

$$\min \sum_{a \in X} y_a + \sum_{b \in Y} y_b$$

s.t.

$$y_a + y_b \geq w_{ab} \quad \forall (a,b) \in E$$

Why? Because],

Expanding: $\sum x_e w_e \leq (y_a + y_b) x_{ab} + (y_{a'} + y_{b'}) x_{a'b'} + \cdots$ all edges $= \sum_a y_a + \sum_b y_b$

Let M: perfect matching in G.

If $\hat{y}$ is a dual feasible, we have

$$\sum_{e \in M} w_e \leq \sum_a \hat{y}_a + \sum_b \hat{y}_b$$

$\hat{y}$ need not be opt., just $\hat{y}$ is feasible.

But, since dual is a minimization problem, best setting is when $\forall e \in M$, s.t. $e = (a, b)$

$$\left[y_a + y_b = w_e = w_{ab} \right\} \text{ called as "tight" edges}$$

s.t. y is feasible for <u>all</u> edges, not just in the Matching M.]

<u>Goal</u>

<u>Algorithm</u> ~ Primal Dual Method

Remember our goal: All edges in perfect M are tight & all other edges are feasible.

If $w(e) = 1$, setting all $y_a = 1$ & all $y_b = 0$, all edges are tight & all perfect matchings are of same weight ≈ max weight.

<u>How</u> <u>to</u> <u>get</u> <u>initial</u> <u>dual</u> <u>feasible</u>?

Set $y_a = \max\limits_{(a,b')} w_{ab'}$ $\quad \forall a \in X, (a,b') \in E$

$y_b = 0 \quad \forall b \in Y$

This will make few edges tight ≈

for those e, $y_a + y_b = w_{ab} = w_e$

In this tight subgraph, find maximum (unweighted) matching.

If it itself is perfect, we are done.

Otherwise, systematically improve the tight subgraph

[TODO - tomorrow]