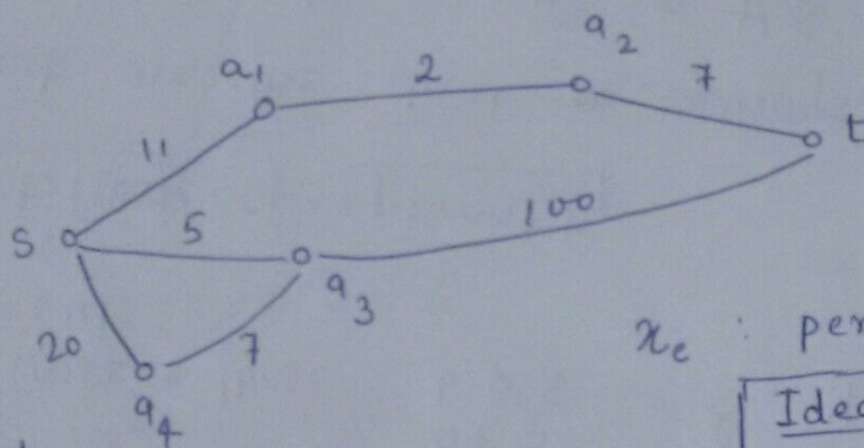
19-9-19

SHORTEST PATH

- LP, proof of optimality

Given: Non-negative edge weighted graph, undirected: $G = (V, E)$, $w(e) \geq 0$, s, t

Goal: shortest path between s & t.



x_e : per edge variable.

Idea: Any feasible s-t path must cross the cut [s-t cut] at least once

Primal

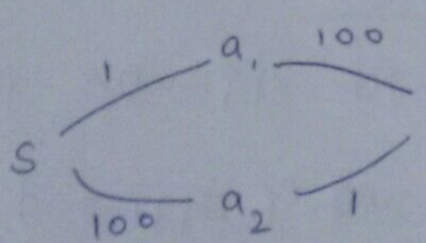
$$\min \sum_{e \in E} x_e w(e) \quad \text{s.t.}$$

$$\sum_{e \in \delta(S)} x_e \geq 1 \quad \text{for all } (S, V \setminus S) \text{ s-t cuts.}$$

ie $S \subseteq V$, $s \in S$, $t \notin S$.

& $x_e \in \{0, 1\}$ [ILP]

Ex:



Cut $S = \{s, a_1\}$ forbids LP to pick s-a_1, a_2-t in the solution as it violates constraint for cut S, $V \setminus S$

★ Now, we have exponentially many constraints.

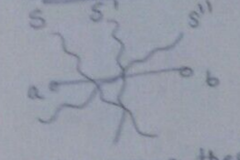
Dual

$\because$ Primal is a minimization so we want lower bound (as large / tight as possible) on primal objective $\sum_e w_e x_e$

1 Dual variable per constraint : $y_S \ \forall S$: cut

obj : max $\sum_{\substack{S \subseteq V, \\ s \in S, t \notin S}} y_S$ s.t.

constraints : [one per x_e ie edge in primal]



(a,b) crosses these cuts

$$\sum_{\substack{S : e = (a,b), \\ a \in S, \\ b \notin S}} y_S \le w_e \quad \forall e \in E$$

& $y_S \geq 0 \ \forall S$

Certificate of optimality :

What are the dual settings that are feasible?

- Trivial setting : $y_S = 0 \ \forall S$
  - ↳ gives feasibility & trivial lower bound on opt ie 0 ✓

What is not a dual feasible for previous ex ?

$S_3 = \{s, a_1, a_2\}$, $S_2 = \{s, a_2\}$, $S_1 = \{s\}$

& set $y_{S_3} = 7$, $y_{S_1} = 11$, $y_{S_2} = 2$

then for edge $s - a_3$ that crosses all 3 cuts

$$\sum_{\substack{S : e \\ \text{crosses} \\ S}} y_S = 11 + 7 + 2 > w_e = 5$$ but we want $\le$.

Starting dual feasible which may be primal infeasible setting

$\equiv y_S = 0 \ \forall S$, $x_e = 0 \ \forall e$.

This gives tight subgraph as empty subgraph.

Let $S_1 = \{s\}$, $V \setminus S_1$ is a cut.

Increase y_{S_1} till for some edge, constraint becomes tight.

i.e. @ $y_{S_1} = 5$, $e = (s - a_3)$ edge constraint is tight.

Now, let $S_1 \leftarrow \{s, a_3\}$, $V \setminus S_1$ is a new cut.

Increase y_{S_1} till some edge becomes tight.

@ $y_{S_1} = 6$, $s - a_1$ edge becomes tight.

[earlier 5 + now 6 = $w_e = 11$]

To be continued →