Recap Primal - dual algorithm 25.9.19

$\vec{y} = 0$, $\vec{x} = 0$

For all edges e, set e = unfrozen.

Pick an unfrozen edge e

Increase y_e s.t. a dual constraint for some $u \in V$ is tight.

Pick u in VC, $x_u = 1$.

Freeze (dual variables for) all edges incident on u

Repeat

- Initial dual : feasible
- At the end of algorithm, dual is feasible.
- At the end of algorithm, primal is feasible.

$$\text{Cost of primal} = \sum_{\substack{u: \\ x_u > 0}} x_u c_u$$

$$= \sum_{u: x_u > 0} \left(\sum_{e \in \delta(u)} y_e \right) x_u \quad \cdots\cdots \because \text{its dual constraint is tight.}$$

$$= \sum_{e \in E} y_e (x_u + x_v) \leq \sum_{\substack{e: E, \\ y_e > 0}} 2y_e$$

i.e. a primal constraint $x_u + x_v \geq 1$, although not tight, is still $2 \geq x_u + x_v$

So, primal cost $\leq \sum_{e: E} 2y_e = 2 *$ dual feasible

$\leq 2 *$ dual optimam $= 2 *$ primal optimam

i.e. primal cost $\leq 2 *$ primal OPT

$\rightarrow$ 2- approximation

General LP :

$$\min \sum_{i=1}^{n} c_i x_i$$

s.t.

$$\sum_{i=1}^{n} a_{ji} x_i \geq b_j, \ j = 1 \text{ to } n$$

$$x_i \geq 0, \ \forall i$$

$$\max \sum_{j=1}^{m} y_j b_j$$

s.t. $\sum_{j=1}^{m} a_{ji} y_j \leq c_i \quad \forall_{i=1..n}$

$y_j \geq 0, \ \forall j$

$$\sum_{i=1}^{n} c_i x_i \geq \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ji} y_j \right) x_i$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji} x_i \right) y_j \geq \sum_{j=1}^{m} b_j y_j$$

$\Rightarrow$ Every primal feasible solution obj $\geq$ Every dual feasible obj

Now, if x, y were opt: x^*, y^* then

$L_1 \geq L_2 = L_3 \geq L_4 \ \dots$ L_k: previous terms

@ strong duality i.e. at x^*, y^*, $L_1 = L_4$

$\Rightarrow L_1 = L_2 = L_3 = L_4$

$L_1 = L_2 \Rightarrow \sum_{i=1}^{n} c_i x_i^* = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ji} y_j^* \right) x_i^*$

$\sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{m} a_{ji} y_j \right) x_i = 0 \Rightarrow P_1 + P_2 + \dots + P_n = 0$ (*)

$P_1 = \left(c_1 - \sum_{j=1}^{m} a_{j1} y_j \right) x_1$, $x_1 \geq 0$, $(\cdot) \geq 0$

$\Rightarrow$ all $P_i \geq 0 \ \forall i = 1$ to n

$\Rightarrow \forall i, P_i = 0$ to get (*)

$\Rightarrow$ if $x_i > 0 \Rightarrow c_i - \sum_{j=1}^{m} a_{ji} y_j = 0$

$\Rightarrow$ dual constraint is tight.

$\Rightarrow$ Dual is slack $\Rightarrow x_i = 0$

Relaxed complementary conditions s.t. we get $(\alpha\text{-}\beta)$-approximation ie $\alpha \cdot \beta$-approximation.

s.t. $\alpha \geq 1$, $\beta \geq 1$ for weighted VC.

PCS: If $x_i > 0$ $(x_i = 1)$, dual constraint is tight ie

$$\sum_{e \in S(u)} y_e = c_u \quad \dots\dots \ \alpha = 1$$

DCS: If $y_e > 0$ then $x_u + x_v \leq 2 \quad \dots \ \beta = 2$

General LP: If $x_i > 0$: dual constraint is tight

If $y_j > 0$: primal constraint need not be tight but

$$\alpha b_j \geq \sum_{i=1}^{n} a_{ji} x_i \geq b_j$$

This gives α-approximation as follows:

$$\sum_{i=1}^{n} x_i c_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ji} y_j \right) x_i \leq \alpha \sum_{j=1}^{m} b_j y_j$$

$= \alpha \sum_{j=1}^{m} b_j y_j = \alpha *$ dual feasible

$\leq \alpha *$ dual opt $= \alpha *$ primal opt

Q: How to get relaxed CS for $(\alpha\text{-}\beta)$-approximation?