DulMage Mendelsohn Decomposition

Here we give a well-known decomposition theorem for bipartite graphs, the Dulmage Mendelsohn Decomposition [2]. The version presented here is from [1].

A maximum matching M in a bipartite graph $G = (\mathcal{A} \cup \mathcal{B}, E)$ has the following important properties: M defines a partition of $\mathcal{A} \cup \mathcal{B}$ into three disjoint sets: odd ($\mathcal{O}$), even ($\mathcal{E}$) and unreachable ($\mathcal{U}$). A node $u \in \mathcal{E}$ (respectively, $\mathcal{O}$) if there is an *even* (*odd*) length alternating path in G from an unmatched node to u. A node $u \in \mathcal{U}$, that is, it is *unreachable*, if there is no alternating path in G from an unmatched node to u.

Lemma: Let $\mathcal{E}$, $\mathcal{O}$, and $\mathcal{U}$ be the sets of nodes defined by M in G. Then,

(a) $\mathcal{E}$, $\mathcal{O}$ and $\mathcal{U}$ are pairwise disjoint, and independent of the maximum matching M.

(b) In any maximum matching of G, every node in $\mathcal{O}$ is matched with a node in $\mathcal{E}$, and every node in $\mathcal{U}$ is matched with another node in $\mathcal{U}$. The size of a maximum matching is $|\mathcal{O}| + |\mathcal{U}|/2$.

(c) No maximum matching of G contains an edge between a node in $\mathcal{O}$ and a node in $\mathcal{O} \cup \mathcal{U}$. G contains no edge between a node in $\mathcal{E}$ and a node in $\mathcal{E} \cup \mathcal{U}$.

Proof

(a) The set $\mathcal{U}$ is disjoint from $\mathcal{E} \cup \mathcal{O}$ by definition. To prove that $\mathcal{E}$ is disjoint from $\mathcal{O}$, assume that a node u is reachable by an *even* length path from a node a and an *odd* length path from a node b. Note that $a \neq b$ since G is bipartite. Composing the two paths, we get an augmenting path in G with respect to M, contradicting the maximality of M.

To prove that this partition in independent of M, let N be any other maximum cardinality matching in G. $M \oplus N$ consists of alternating paths and cycles and each of these paths and cycles are of even length. Since G is bipartite it is clear that the cycles are of even length. For paths, assume that a path has more edges from N, then such a path is an augmenting path w.r.t. M, a contradiction to maximality of M. A similar argument holds if there are more edges from M. Using these paths and cycles to switch from M to N does not alter the *odd/even/unreachable* status of nodes, hence the partition is independent of the maximum cardinality matching.

(b) If a matched node u is reachable from a free node by an *odd* length path with respect to any maximum cardinality matching, then its partner is reachable by an *even* length path. Thus, all edges in any maximum matching of G are either $\mathcal{O}\mathcal{E}$ or $\mathcal{U}\mathcal{U}$ edges. Further, any node in $\mathcal{U}$ must be matched by a maximum matching, for, if not, the node is reachable with an *even* length (zero length) path from itself. Also a node in $\mathcal{O}$ must be matched by a maximum matching since an *odd* length alternating path starting and ending with a free node is an augmenting path. Thus, the size of any maximum matching is $|\mathcal{O}| + |\mathcal{U}|/2$.

(c) Nodes in $\mathcal{E}$ are reachable by an alternating path from an unmatched node. Such paths end in a matching edge. If there were an edge between two nodes in $\mathcal{E}$, we could use that to construct an augmenting path, contradicting maximality. Finally, if there were an edge between a node in $\mathcal{E}$ and a node in $\mathcal{U}$, such an edge would be a non-matching edge. We could use that to extend the alternating path and reach the node in $\mathcal{U}$, a contradiction to the definition of nodes in $\mathcal{U}$.

This finishes the proof of the lemma.

References

[1] R.W. Irving, T. Kavitha, K. Mehlhorn, D. Michail, and K. Paluch. Rank-maximal matchings. *ACM Transactions on Algorithms*, 2(4):602–610, 2006.

[2] L. Lovász and M. D. Plummer. Matching Theory. *North-Holland*, (Mathematics Studies 121), 1986.