Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution:

Concept: Linear Combinations

2. (2 points) Consider the vectors $[x, y]$, $[a, b]$ and $[c, d]$.

   (a) Express $[x, y]$ as a linear combination of $[a, b]$ and $[c, d]$.

   Solution:

   (b) Based on the expression that you have derived above, write down the condition under which $[x, y]$ cannot be expressed as a linear combination of $[a, b]$ and $[c, d]$. (Must: the condition should talk about some relation between the scalars a, b, c, d, x and y)

   Solution:

Concept: Elementary matrices

3. (1 point) Consider the matrix $E_{r+\alpha q}$ that represents the elementary row operation of adding a multiple of α times row q to row r.

Under what conditions is $E_{r+\alpha q}$

   (a) upper triangular?

   Solution:

   (b) lower triangular?

   Solution:

4. (1 point) Let $E_1, E_2, E_3, \ldots, E_n$ be n elementary matrices. Let $(i_1, j_1), (i_2, j_2), \ldots (i_n, j_n)$ be the position of the non-zero off-diagonal element in each of these elementary matrices. Further, *if* $k \neq m$ *then* $(i_k, j_k) \neq (i_m, j_m)$ (*i.e.*, no two elementary matrices in the sequence have a non-zero off-diagonal element in the same position). Prove that the product of these n elementary matrices will have all diagonal entries as 1. (Proving this will help you understand why the diagonal elements of L are always equal to 1.)

Solution:

Concept: Inverse

5. (½ point) If A is a square invertible matrix then prove that the inverse of $A^\top$ is $A^{-1\top}$

Solution:

6. (2 points) Prove that a $n \times n$ matrix A is invertible if and only if Gaussian Elimination of A produces n non-zero pivots.

Solution:

Proof (the if part):

Proof (the only if part):

7. (1 point) If A is a $n \times n$ matrix then what is the cost of:
   (a) Computing A^{-1}

   Solution:

   (b) Computing $A^{-1}b$

   Solution:

Concept: LU factorisation

8. (1 ½ points) In the lecture, we saw that once we do LU factorisation, we can solve $A\mathbf{x} = \mathbf{b}$ by solving two triangular systems $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$.

(a) Prove that $L\mathbf{c} = \mathbf{b}$.

Solution:

(b) What is the cost of solving a triangular system (say $L\mathbf{c} = \mathbf{b}$ or $U\mathbf{x} = \mathbf{c}$)?

Solution:

(c) Based on the above results can you comment on the utility of LU factorisation?

Solution:

One time cost of LU factorisation:

Recurring cost of solving $L\mathbf{c} = \mathbf{b}$ *and* $U\mathbf{x} = \mathbf{c}$:

Hence, ...

9. (2 points) Consider the following system of linear equations. Find the LU factorisation of the matrix A corresponding to this system of linear equations. Show all the steps involved. (this is where you will see what happens when you have to do more than 1 permutations).

$$\begin{aligned} x + y - 2z &= -3 \\ w + 2x - y &= +2 \\ w - 4x - 7y - z &= -19 \\ 2w + 4x + y - 3z &= -2 \end{aligned}$$

Solution:

10. (1 ½ points) For a square matrix A:

(a) Prove or disprove: LU factorisation is unique.

Solution:

(b) Prove or disprove: LDU factorisation is unique.

Solution:

11. (1 ½ points) Consider the matrix A which factorises as:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Without computing A or A^{-1} argue that

(a) A is invertible (I am looking for an argument which relies on a fact about elementary matrices)

Solution:

(b) A is symmetric (convince me that $A_{ij} = A_{ji}$ without computing A)

Solution:

(c) A is tridiagonal (again, without computing A convince me that all elements except along the 3 diagonals will be 0.)

Solution:

Concept: Lines and planes

12. (1 ½ points) Consider the following system of linear equations

$$\begin{aligned} a_1x_1 + b_1y_1 + c_1z_1 &= 1 \\ a_2x_2 + b_2y_2 + c_2z_2 &= 0 \\ a_3x_3 + b_3y_3 + c_3z_3 &= -1 \end{aligned}$$

Each equation represents a plane, so find out the values for the coefficients such that the following conditions are satisfied:

1. All planes intersect at a line
2. All planes intersect at a point
3. Every pair of planes intersects at a different line.

Solution:

13. (1 ½ points) Starting with a first plane $u + 2v - w = 6$, find the equation for

(a) the parallel plane through the origin.

Solution:

(b) a second plane that also contains the points (6,0,0) and (2,2,0).

Solution:

(c) a third plane that meets the first and second in the point (4, 1, 0).

Solution:

Concept: Transpose

14. (2 points) Consider the transpose operation.

(a) Show that it is a linear transformation.

Solution:

(b) Find the matrix corresponding to this linear transformation.

Solution: