CS7015 (Deep Learning) : Lecture 23

Generative Adversarial Networks (GANs)

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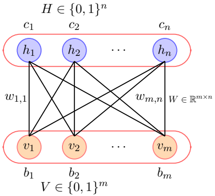
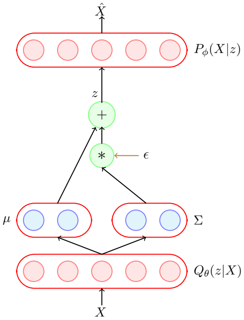
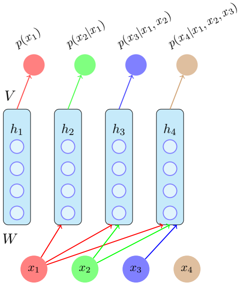
Module 23.1: Generative Adversarial Networks - The intuition

- So far we have looked at generative models which explicitly model the joint probability distribution or conditional probability distribution







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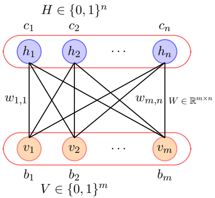

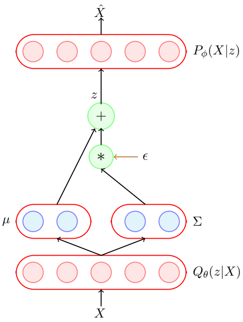

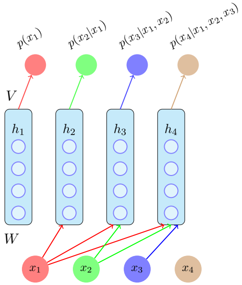

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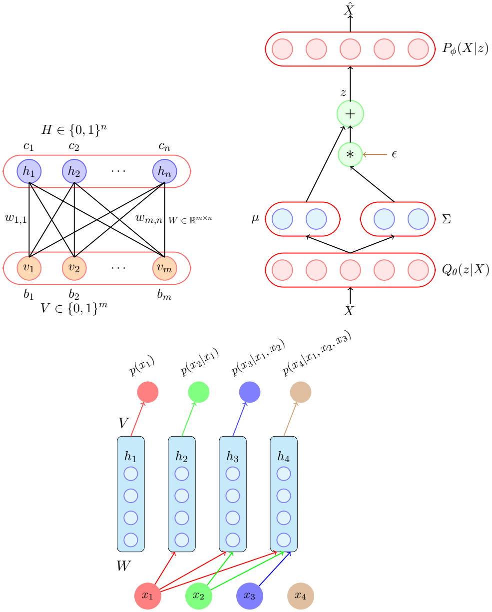

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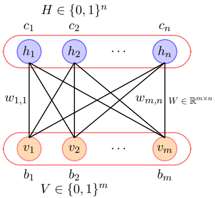

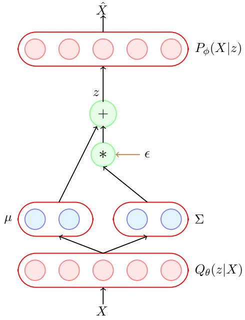

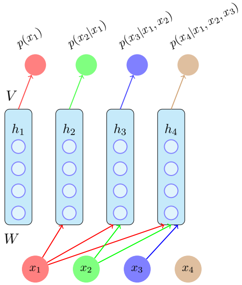

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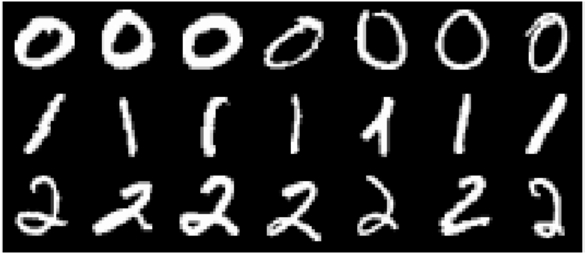
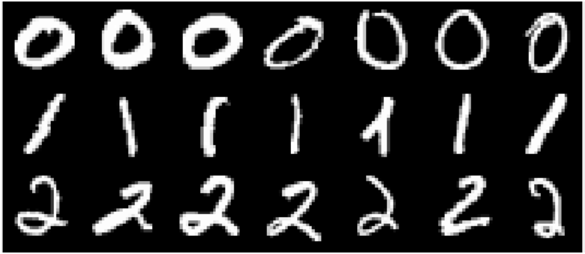


- As usual we are given some training data (say, MNIST images) which obviously comes from some underlying distribution



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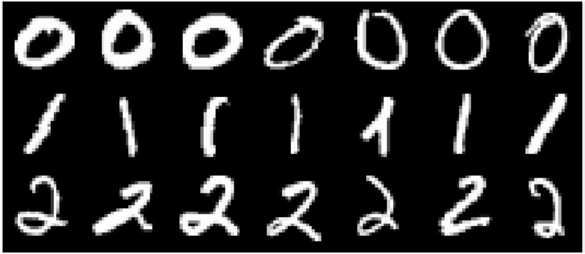

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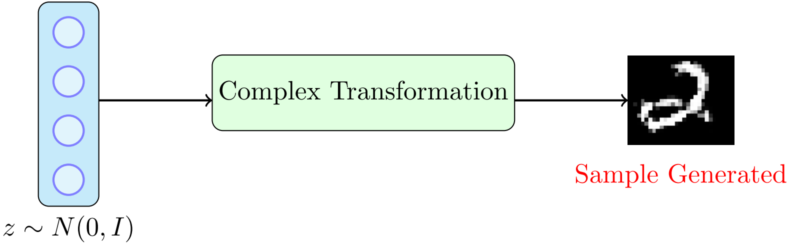
- In other words, we want to sample from a complex high dimensional distribution which is intractable (recall RBMs, VAEs and AR models deal with this intractability in their own way)



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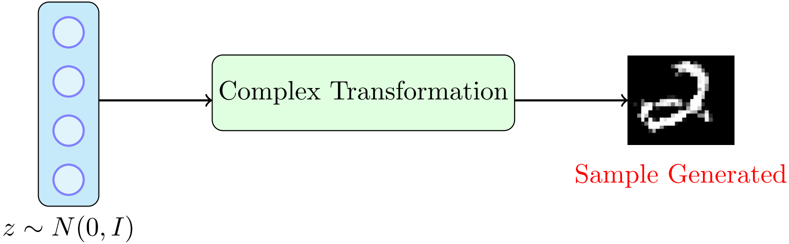

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- In other words, we will take a $z \sim N(0, I)$, learn to make a series of complex transformations on it so that the output looks as if it came from our training distribution

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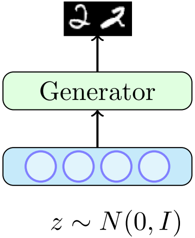

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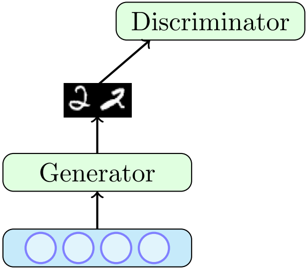
- There are two players in the game: a generator



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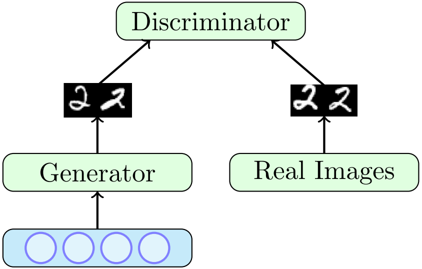

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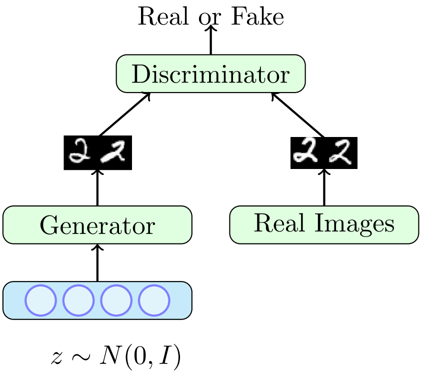

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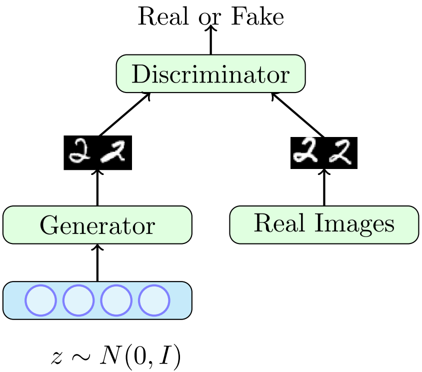
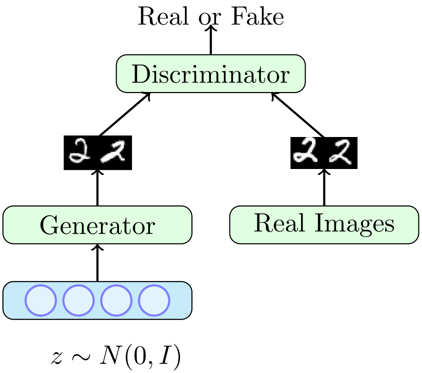
- The job of the discriminator is to get better and better at distinguishing between true images and generated (fake) images

- So let's look at the full picture





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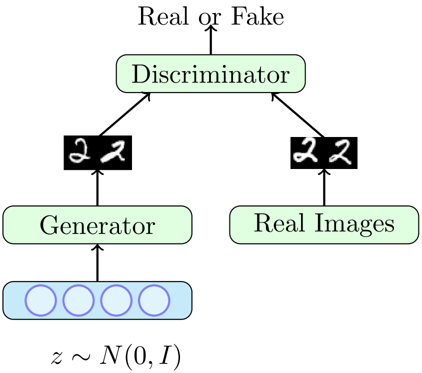
- Let G_ϕ be the generator and D_θ be the discriminator (ϕ and θ are the parameters of G and D, respectively)



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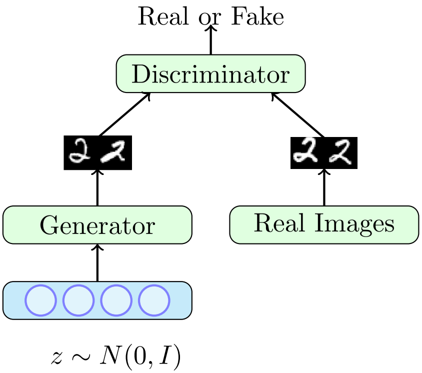
- Let G_ϕ be the generator and D_θ be the discriminator (ϕ and θ are the parameters of G and D, respectively)
- We have a neural network based generator which takes as input a noise vector $z \sim N(0, I)$ and produces $G_\phi(z) = X$



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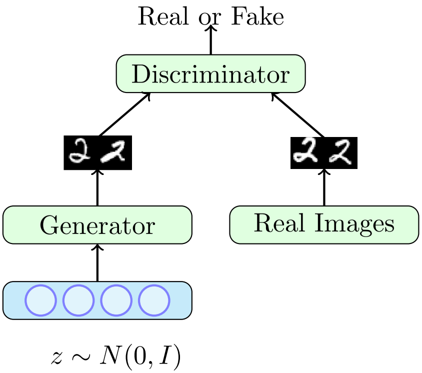
- Let G_ϕ be the generator and D_θ be the discriminator (ϕ and θ are the parameters of G and D, respectively)
- We have a neural network based generator which takes as input a noise vector $z \sim N(0, I)$ and produces $G_\phi(z) = X$
- We have a neural network based discriminator which could take as input a real X or a generated $X = G_\phi(z)$ and classify the input as real/fake



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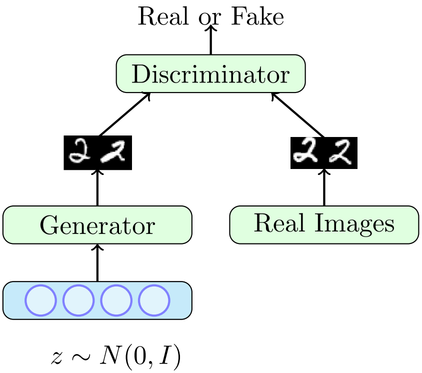

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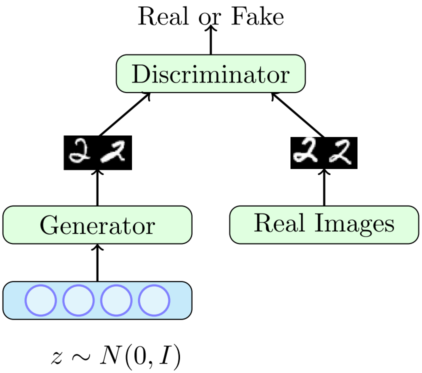

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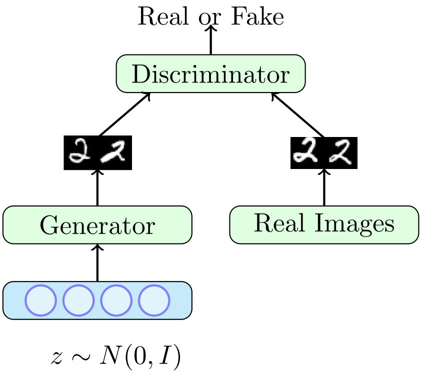

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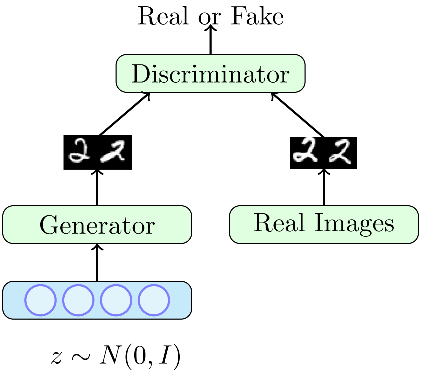

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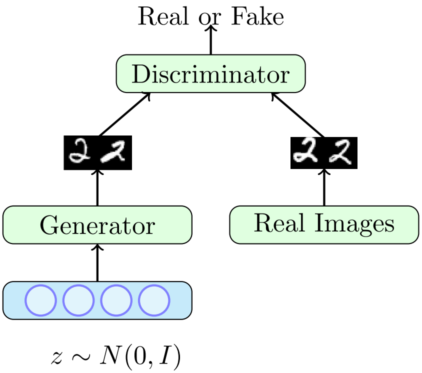
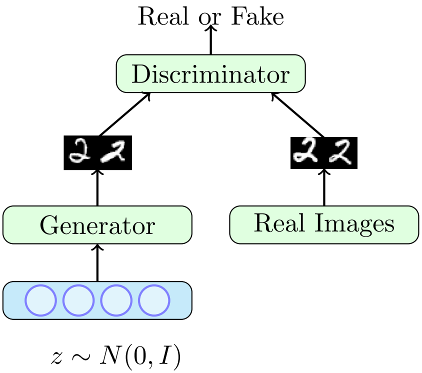
- This score will be between 0 and 1 and will tell us the probability of the image being real or fake
- For a given z, the generator would want to maximize $\log D_\theta(G_\phi(z))$ (log likelihood) or minimize $\log(1 - D_\theta(G_\phi(z)))$

- This is just for a single z and the generator would like to do this for all possible values of z,





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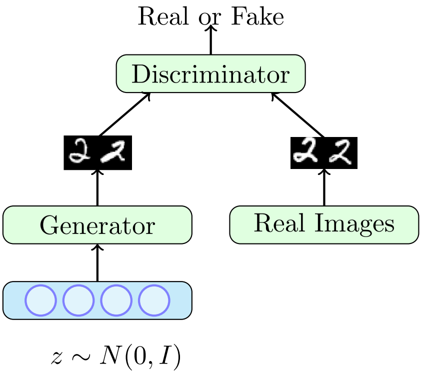
- For example, if z was discrete and drawn from a uniform distribution (*i.e.*, $p(z) = \frac{1}{N}\ \forall z$) then the generator's objective function would be

$$\min_{\phi} \sum_{i=1}^{N} \frac{1}{N} \log(1 - D_\theta(G_\phi(z)))$$



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- However, in our case, z is continuous and not uniform ($z \sim N(0, I)$) so the equivalent objective function would be

$$\min_{\phi} \int p(z) \log(1 - D_\theta(G_\phi(z)))$$

$$\min_{\phi} E_{z \sim p(z)}[\log(1 - D_\theta(G_\phi(z)))]$$

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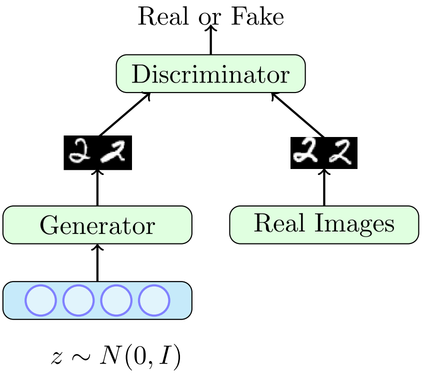

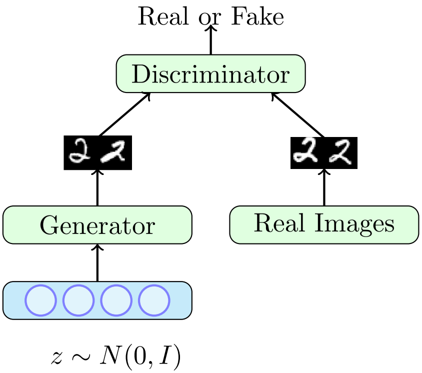

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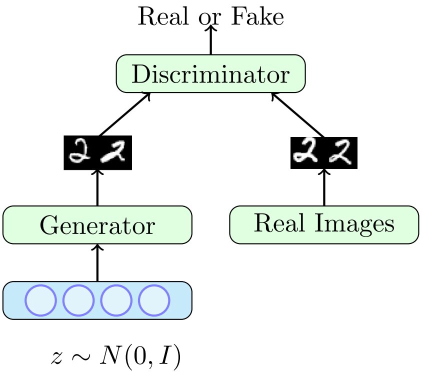

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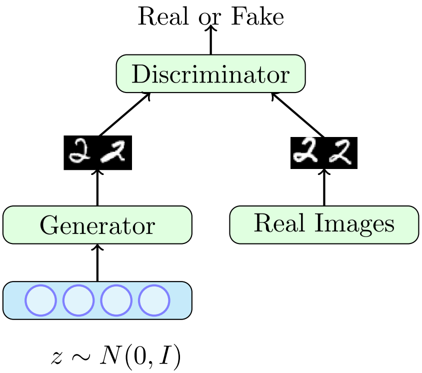

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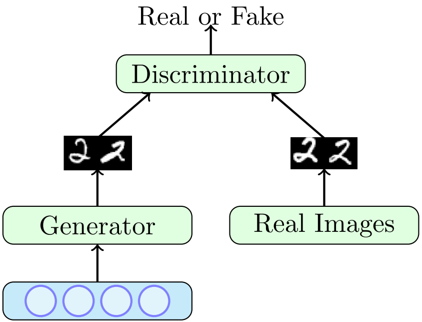
- And it should do this for all possible real images and all possible fake images
- In other words, it should try to maximize the following objective function

$$\max_{\theta} E_{x \sim p_{data}}[\log D_\theta(x)] + E_{z \sim p(z)}[\log(1 - D_\theta(G_\phi(z)))]$$



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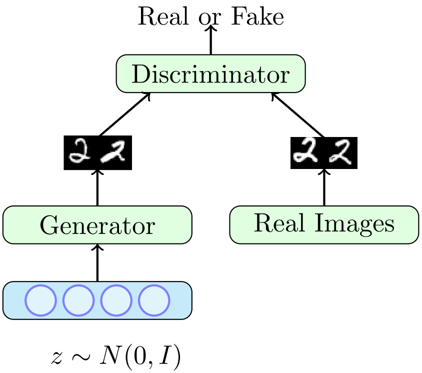

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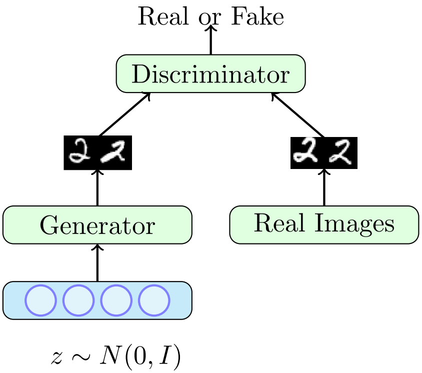

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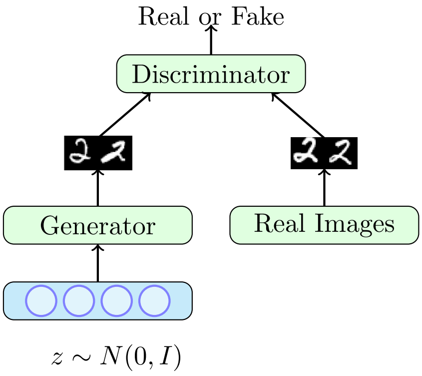

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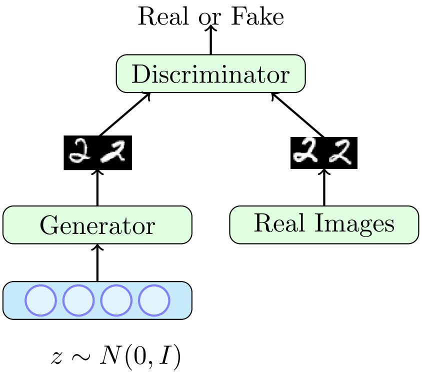
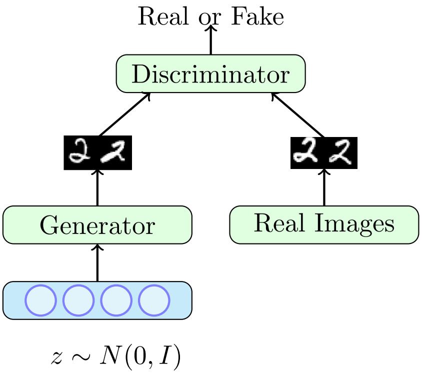
- The first term in the objective is only w.r.t. the parameters of the discriminator (θ)
- The second term in the objective is w.r.t. the parameters of the generator (ϕ) as well as the discriminator (θ)
- The discriminator wants to maximize the second term whereas the generator wants to minimize it (hence it is a two-player game)

- So the overall training proceeds by alternating between these two step





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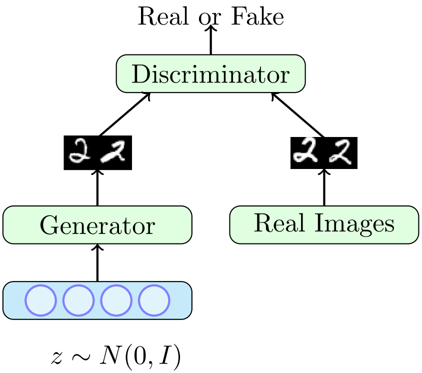

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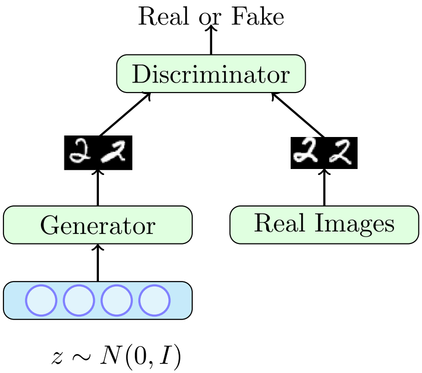

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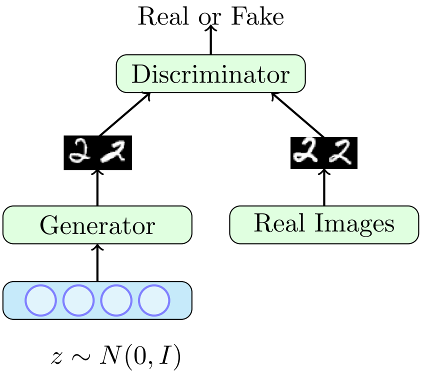

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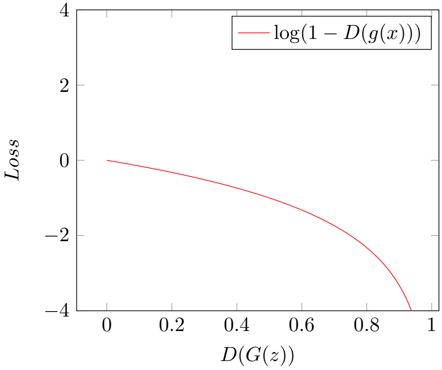
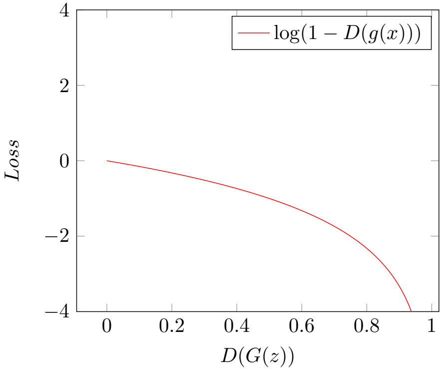


- When the sample is likely fake, we want to give a feedback to the generator (using gradients)



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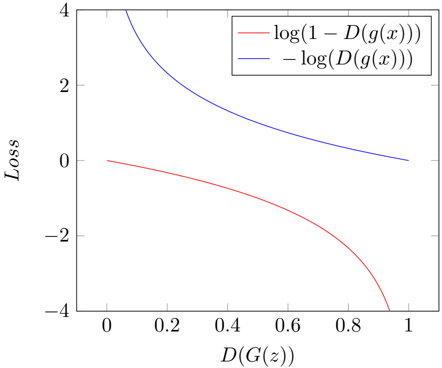

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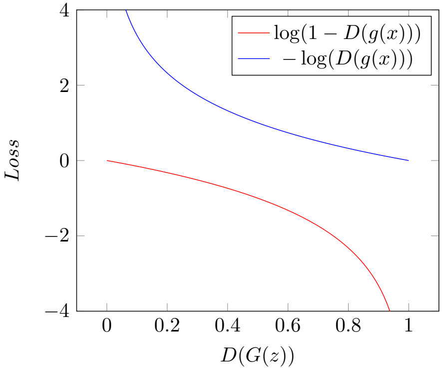

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- Trick: Instead of minimizing the likelihood of the discriminator being correct, maximize the likelihood of the discriminator being wrong
- In effect, the objective remains the same but the gradient signal becomes better

With that we are now ready to see the full algorithm for training GANs

---

procedure GAN TRAINING

end procedure

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```
procedure GAN Training
    for number of training iterations do

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```
procedure GAN TRAINING
    for number of training iterations do
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procedure GAN TRAINING
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$$\nabla_\phi \frac{1}{m} \sum_{i=1}^{m} \left[\log\left(D_\theta\left(G_\phi\left(z^{(i)}\right)\right)\right)\right]$$

  end for

end procedure

Module 23.2: Generative Adversarial Networks - Architecture

- We will now look at one of the popular neural networks used for the generator and discriminator (Deep Convolutional GANs)

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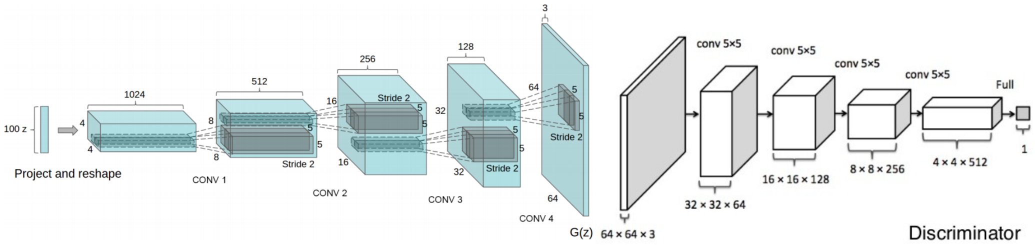

Figure: Generator (Redford et al 2015) (left) and discriminator (Yeh et al 2016) (right) used in DCGAN

Architecture guidelines for stable Deep Convolutional GANs

- Replace any pooling layers with strided convolutions (discriminator) and fractional-strided convolutions (generator).

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- Use LeakyReLU activation in the discriminator for all layers

Module 23.3: Generative Adversarial Networks - The Math Behind it

- We will now delve a bit deeper into the objective function used by GANs and see what it implies

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- Can we prove this formally even though the model is not explicitly computing this density?
- We will try to prove this over the next few slides

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Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_{D} V(G, D)$ is achieved **if** $p_G = p_{data}$

The 'only if' part: The global minimum of the virtual training criterion $C(G) = \max_{D} V(G, D)$ is achieved **only if** $p_G = p_{data}$

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The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **if** $p_G = p_{data}$

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(c) Show that $a < b \ \forall \ p_G \neq p_{data}$(and hence the minimum $V(D, G)$ is achieved when $p_G = p_{data}$)

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- First let us look at the objective function again

$$\min_{\phi} \max_{\theta} \quad [\mathbb{E}_{x \sim p_{data}} \log D_\theta(x) + \mathbb{E}_{z \sim p(z)} \log(1 - D_\theta(G_\phi(z)))]$$

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- The above replacement follows from the *law of the unconscious statistician* (click to link of wikipedia page)

- Okay, so our revised objective is given by

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$$\frac{d}{d(D_\theta(x))} \left(p_{data}(x) \log D_\theta(x) + p_G(x) \log(1 - D_\theta(x))\right) = 0$$

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$$\frac{d}{d(D_\theta(x))}\left(p_{data}(x) \log D_\theta(x) + p_G(x) \log(1 - D_\theta(x))\right) = 0$$

$$p_{data}(x)\frac{1}{D_\theta(x)} + p_G(x)\frac{1}{1 - D_\theta(x)}(-1) = 0$$

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p_{data}(x) \frac{1}{D_\theta(x)} + p_G(x) \frac{1}{1 - D_\theta(x)}(-1) &= 0 \\
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(p_{data}(x))(1 - D_\theta(x)) &= (p_G(x))(D_\theta(x)) \\
D_\theta(x) &= \frac{p_{data}(x)}{p_G(x) + p_{data}(x)}
\end{aligned}$$

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$$\begin{aligned} V(G, D_G^*) &= \int_x p_{data}(x) \log D(x) + p_G(x) \log\left(1 - D(x)\right) dx \\ &= \int_x p_{data}(x) \log \frac{1}{2} + p_G(x) \log\left(1 - \frac{1}{2}\right) dx \\ &= \log 2 \int_x p_G(x) dx - \log 2 \int_x p_{data}(x) dx \end{aligned}$$

- This means for any given generator

$$D_G^*(G(x)) = \frac{p_{data}(x)}{p_{data}(x) + p_G(x)}$$

- Now the if part of the theorem says "if $p_G = p_{data}$ ...."
- So let us substitute $p_G = p_{data}$ into $D_G^*(G(x))$ and see what happens to the loss functions

$$D_G^* = \frac{p_{data}}{p_{data} + p_G} = \frac{1}{2}$$

$$\begin{aligned} V(G, D_G^*) &= \int_x p_{data}(x) \log D(x) + p_G(x) \log (1 - D(x))\, dx \\ &= \int_x p_{data}(x) \log \frac{1}{2} + p_G(x) \log \left(1 - \frac{1}{2}\right) dx \\ &= \log 2 \int_x p_G(x) dx - \log 2 \int_x p_{data}(x) dx \\ &= -2 \log 2 \qquad = -\log 4 \end{aligned}$$

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **if** $p_G = p_{data}$

(a) Find the value of $V(D, G)$ when the generator is optimal *i.e.*, when $p_G = p_{data}$

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(c) Show that $a < b \ \forall \ p_G \neq p_{data}$(and hence the minimum $V(D, G)$ is achieved when $p_G = p_{data}$)

The 'only if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **only if** $p_G = p_{data}$

- Show that when $V(D, G)$ is minimum then $p_G = p_{data}$

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- So what we have proved so far is that if the generator is optimal ($p_G = p_{data}$) the discriminator's loss value is $-\log 4$
- We still haven't proved that this is the minima
- For example, it is possible that for some $p_G \neq p_{data}$, the discriminator's loss value is lower than $-\log 4$
- To show that the discriminator achieves its lowest value "if $p_G = p_{data}$", we need to show that for all other values of p_G the discriminator's loss value is greater than $-\log 4$

- To show this we will get rid of the assumption that $p_G = p_{data}$

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$$C(G) = \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) + p_G(x) \log \left(1 - \frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) \right] dx$$

- To show this we will get rid of the assumption that $p_G = p_{data}$

$$
\begin{aligned}
C(G) &= \int_x \left[p_{data}(x) \log\left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) + p_G(x) \log\left(1 - \frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) \right] dx \\
&= \int_x \left[p_{data}(x) \log\left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) + p_G(x) \log\left(\frac{p_G(x)}{p_G(x) + p_{data}(x)} \right) + (\log 2 - \log 2)(p_{data} + p_G) \right] dx
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&= \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) + p_G(x) \log \left(\frac{p_G(x)}{p_G(x) + p_{data}(x)} \right) + (\log 2 - \log 2)(p_{data} + p_G) \right] dx \\
&= -\log 2 \int_x (p_G(x) + p_{data}(x))\, dx \\
&\quad + \int_x \left[p_{data}(x) \left(\log 2 + \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) \right) + p_G(x) \left(\log 2 + \log \left(\frac{p_G(x)}{P p_G(x) + p_{data}(x)} \right) \right) \right] dx
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&= -\log 2(1+1) \\
&\quad + \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{\frac{p_G(x)+p_{data}(x)}{2}} \right) + p_G(x) \log \left(\frac{p_G(x)}{\frac{p_G(x)+p_{data}(x)}{2}} \right) \right] dx
\end{aligned}
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&= -\log 2 \int_x (p_G(x) + p_{data}(x))\, dx \\
&\quad + \int_x \left[p_{data}(x) \left(\log 2 + \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) \right) + p_G(x) \left(\log 2 + \log \left(\frac{p_G(x)}{P p_G(x) + p_{data}(x)} \right) \right) \right] dx \\
&= -\log 2(1+1) \\
&\quad + \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{\frac{p_G(x)+p_{data}(x)}{2}} \right) + p_G(x) \log \left(\frac{p_G(x)}{\frac{p_G(x)+p_{data}(x)}{2}} \right) \right] dx \\
&= -\log 4 + KL \left(p_{data} \| \frac{p_G(x) + p_{data}(x)}{2} \right) + KL \left(p_G \| \frac{p_G(x) + p_{data}(x)}{2} \right)
\end{aligned}
$$

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **if** $p_G = p_{data}$

(a) Find the value of $V(D, G)$ when the generator is optimal *i.e.*, when $p_G = p_{data}$

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- Okay, so we have

$$C(G) = -\log 4 + KL\left(p_{data}||\frac{p_{data}+p_g}{2}\right) + KL\left(p_G||\frac{p_{data}+p_G}{2}\right)$$

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- We know that KL divergence is always ≥ 0

$$\therefore C(G) \geq -\log 4$$

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- Hence the minimum possible value of $C(G)$ is $-\log 4$
- But this is the value that $C(G)$ achieves when $p_G = p_{data}$ (and this is exactly what we wanted to prove)
- We have, thus, proved the **if part** of the theorem

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **if** $p_G = p_{data}$

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$$KL\left(p_{data} \| \frac{p_{data} + p_g}{2}\right) + KL\left(p_G \| \frac{p_{data} + p_G}{2}\right) = 0$$

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$$KL\left(p_{data} \| \frac{p_{data} + p_g}{2}\right) + KL\left(p_G \| \frac{p_{data} + p_G}{2}\right) = JSD(p_{data} \| p_G)$$

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which is minimum only when $p_G = p_{data}$

Module 23.4: Generative Adversarial Networks - Some Cool Stuff and Applications

- In each row the first image was generated by the network by taking a vector z_1 as the input and the last images was generated by a vector z_2 as the input

- In each row the first image was generated by the network by taking a vector z_1 as the input and the last images was generated by a vector z_2 as the input
- All intermediate images were generated by feeding z's which were obtained by interpolating z_1 and z_2 $(z = \lambda z_1 + (1 - \lambda) z_2)$

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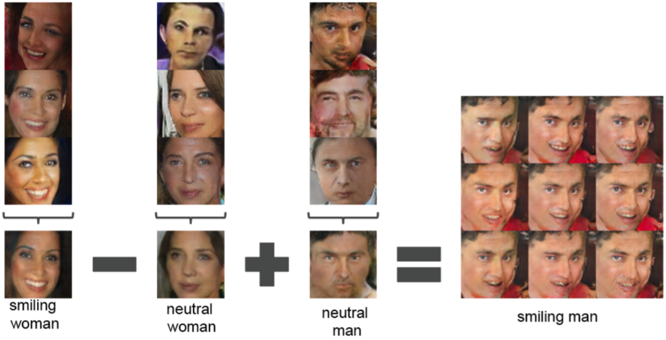
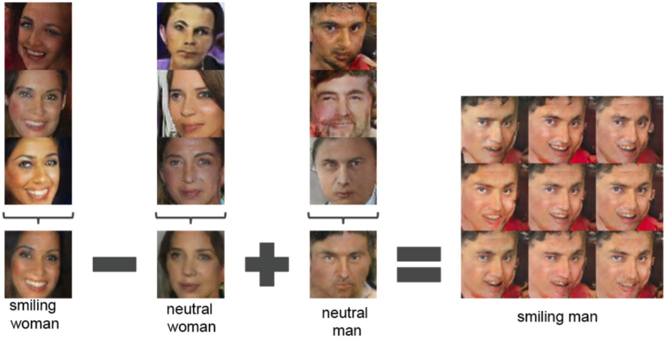
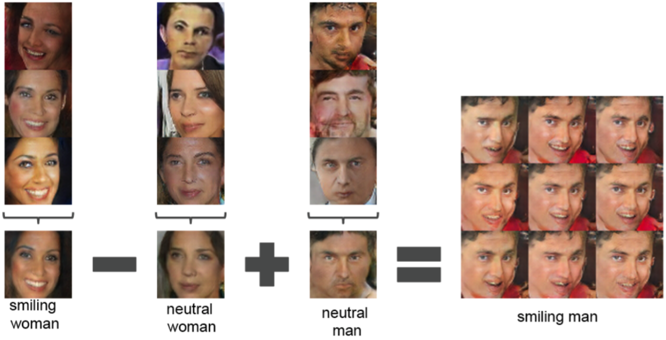
- All intermediate images were generated by feeding z's which were obtained by interpolating z_1 and z_2 ($z = \lambda z_1 + (1 - \lambda) z_2$)
- As we transition from z_1 to z_2 in the input space there is a corresponding smooth transition in the image space also



- The first 3 images in the first column were generated by feeding some z_{11}, z_{12}, z_{13} respectively as the input to the generator



- The first 3 images in the first column were generated by feeding some z_{11}, z_{12}, z_{13} respectively as the input to the generator
- The fourth image was generated by taking an average of $z_1 = z_{11}, z_{12}, z_{13}$ and feeding it to the generator



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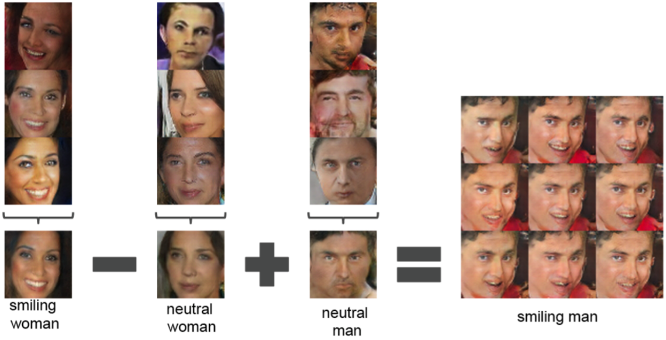
- Similarly we obtain the average vectors z_2 and z_3 for the 2nd and 3rd columns



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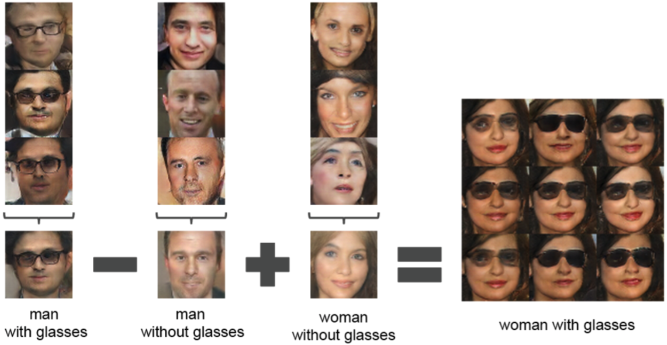
- Similarly we obtain the average vectors z_2 and z_3 for the 2nd and 3rd columns
- If we do a simple vector arithmetic on these averaged vectors then we see the corresponding effect in the generated images



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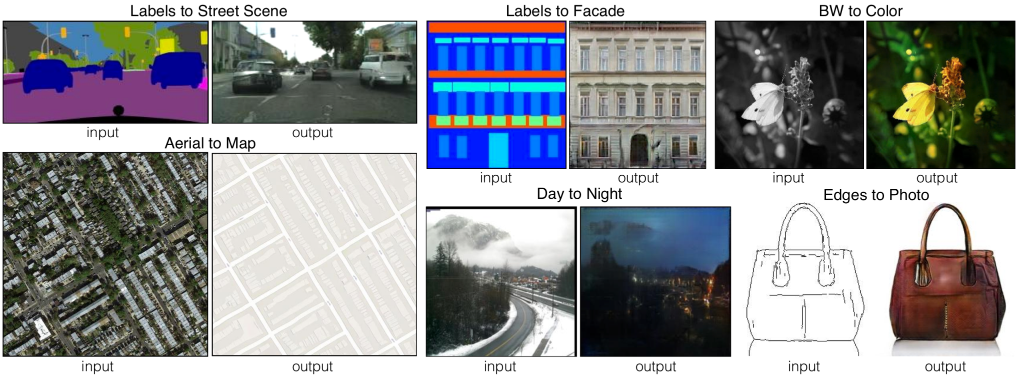

Phillip Isola, Jun-Yan Zhu, Tinghui Zhou, Alexei A. Efros, Image-to-Image Translation with Conditional Adversarial Networks, CVPR, 2017.

Module 23.5: Bringing it all together (the deep generative summary)

	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No

Table: Comparison of Generative Models

	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No
Generation	Yes	Yes	Yes	Yes

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	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No
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Compute P(X)	Intractable	Intractable	Tractable	No

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Compute P(X)	Intractable	Intractable	Tractable	No
Sampling	MCMC	Fast	Slow	Fast

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	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No
Generation	Yes	Yes	Yes	Yes
Compute P(X)	Intractable	Intractable	Tractable	No
Sampling	MCMC	Fast	Slow	Fast
Type of GM	Undirected	Directed	Directed	Directed

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	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No
Generation	Yes	Yes	Yes	Yes
Compute P(X)	Intractable	Intractable	Tractable	No
Sampling	MCMC	Fast	Slow	Fast
Type of GM	Undirected	Directed	Directed	Directed
Loss	KL-divergence	KL-divergence	KL-divergence	JSD

Table: Comparison of Generative Models

	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No
Generation	Yes	Yes	Yes	Yes
Compute P(X)	Intractable	Intractable	Tractable	No
Sampling	MCMC	Fast	Slow	Fast
Type of GM	Undirected	Directed	Directed	Directed
Loss	KL-divergence	KL-divergence	KL-divergence	JSD
Assumptions	X independent given z	X independent given z	None	N.A.

Table: Comparison of Generative Models

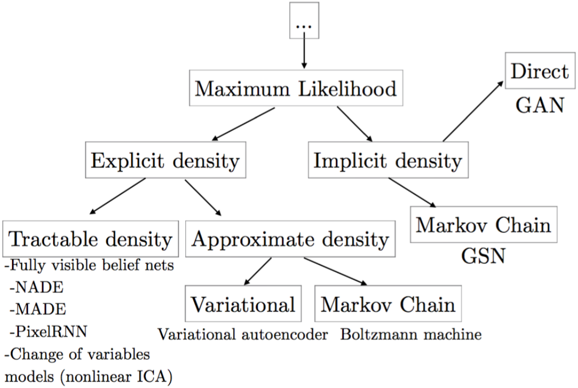
	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No
Generation	Yes	Yes	Yes	Yes
Compute P(X)	Intractable	Intractable	Tractable	No
Sampling	MCMC	Fast	Slow	Fast
Type of GM	Undirected	Directed	Directed	Directed
Loss	KL-divergence	KL-divergence	KL-divergence	JSD
Assumptions	X independent given z	X independent given z	None	N.A.
Samples	Bad	Ok	Good	Good (best)

Table: Comparison of Generative Models

	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No
Generation	Yes	Yes	Yes	Yes
Compute P(X)	Intractable	Intractable	Tractable	No
Sampling	MCMC	Fast	Slow	Fast
Type of GM	Undirected	Directed	Directed	Directed
Loss	KL-divergence	KL-divergence	KL-divergence	JSD
Assumptions	X independent given z	X independent given z	None	N.A.
Samples	Bad	Ok	Good	Good (best)

Table: Comparison of Generative Models

Recent works look at combining these methods: e.g. Adversarial Autoencoders (Makhzani 2015), PixelVAE (Gulrajani 2016) and PixelGAN Autoencoders (Makhzani 2017)



Source: Ian Goodfellow, NIPS 2016 Tutorial: Generative Adversarial Networks