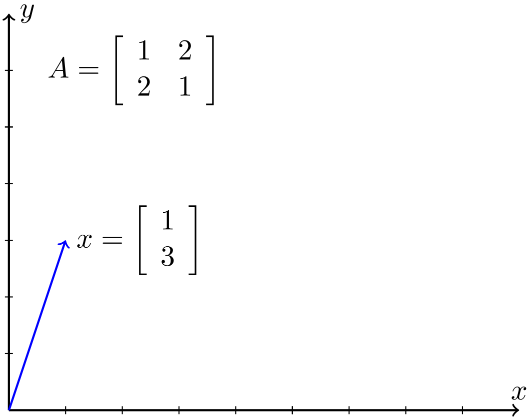
CS7015 (Deep Learning) : Lecture 6

Eigen Values, Eigen Vectors, Eigen Value Decomposition, Principal Component Analysis, Singular Value Decomposition

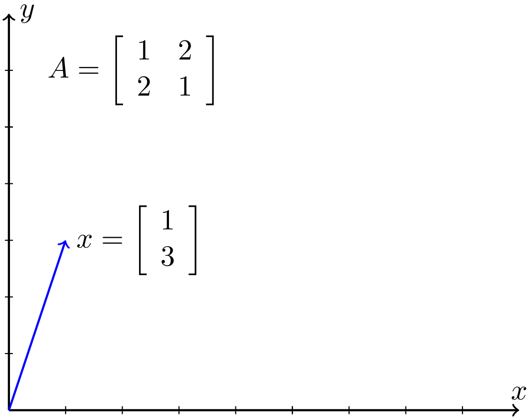
Prof. Mitesh M. Khapra

Department of Computer Science and Engineering
Indian Institute of Technology Madras

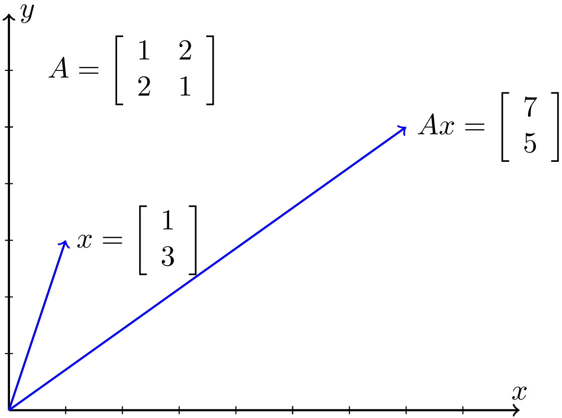
Module 6.1 : Eigenvalues and Eigenvectors



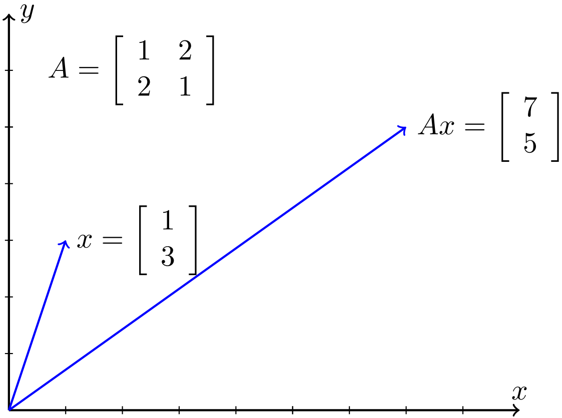
- What happens when a matrix hits a vector?



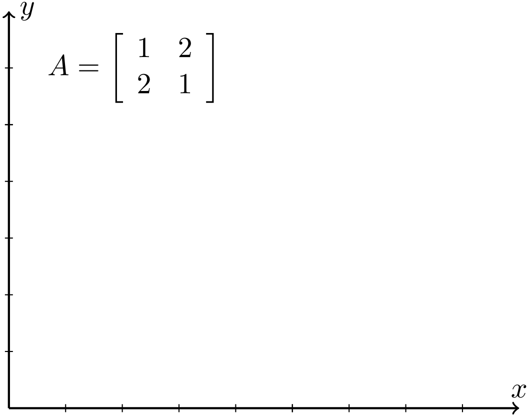
- What happens when a matrix hits a vector?
- The vector gets transformed into a new vector (it strays from its path)



- What happens when a matrix hits a vector?
- The vector gets transformed into a new vector (it strays from its path)

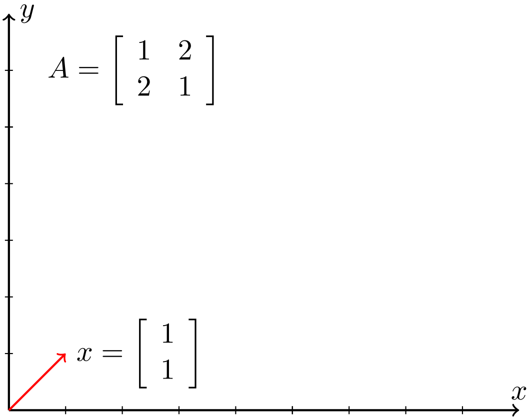


- What happens when a matrix hits a vector?
- The vector gets transformed into a new vector (it strays from its path)
- The vector may also get scaled (elongated or shortened) in the process.

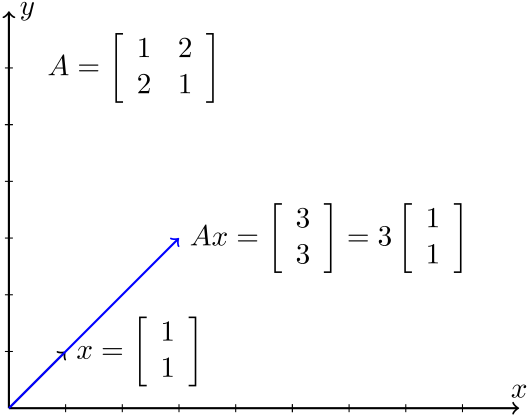


$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

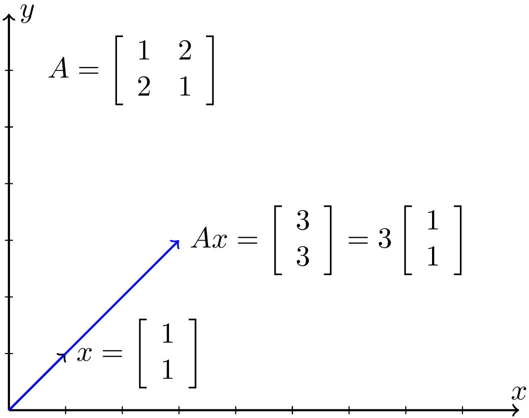
- For a given square matrix A, there exist special vectors which refuse to stray from their path.



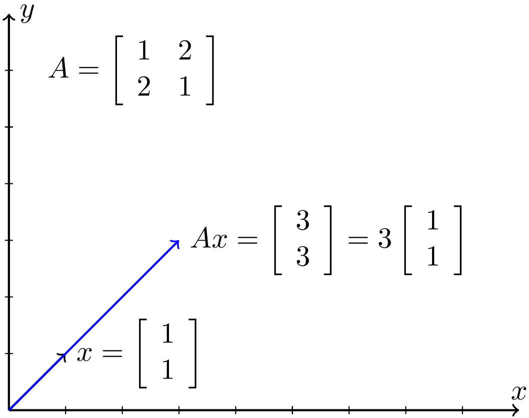
- For a given square matrix A, there exist special vectors which refuse to stray from their path.



- For a given square matrix A, there exist special vectors which refuse to stray from their path.

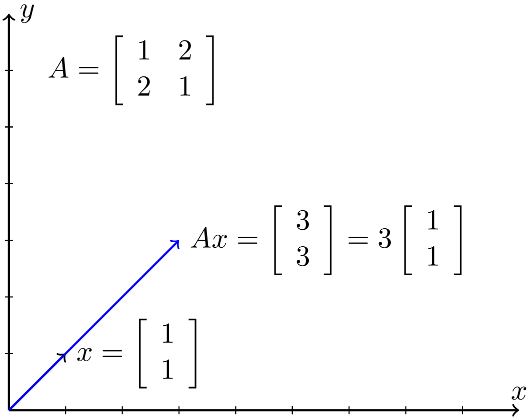


- For a given square matrix A, there exist special vectors which refuse to stray from their path.
- These vectors are called eigenvectors.



- For a given square matrix A, there exist special vectors which refuse to stray from their path.
- These vectors are called eigenvectors.
- More formally,

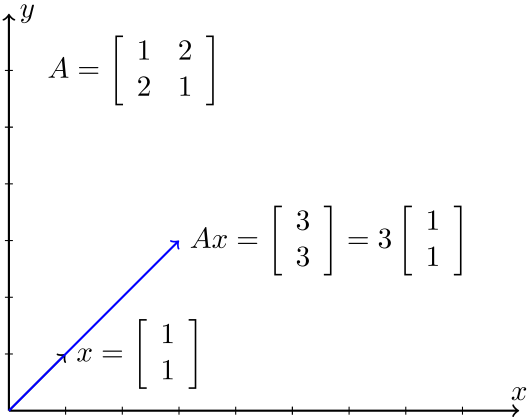
$$Ax = \lambda x \text{ [direction remains the same]}$$



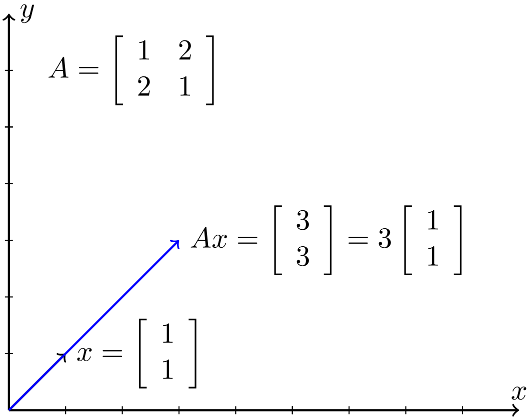
- For a given square matrix A, there exist special vectors which refuse to stray from their path.
- These vectors are called eigenvectors.
- More formally,

$$Ax = \lambda x \text{ [direction remains the same]}$$

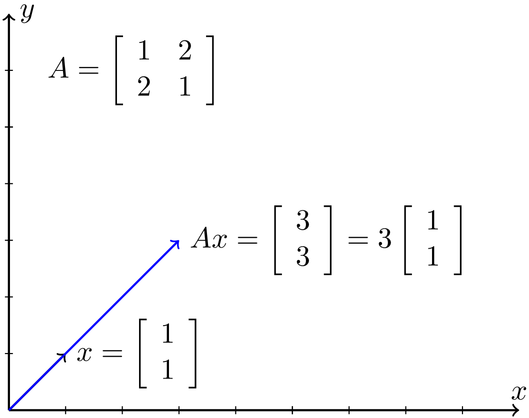
- The vector will only get scaled but will not change its direction.



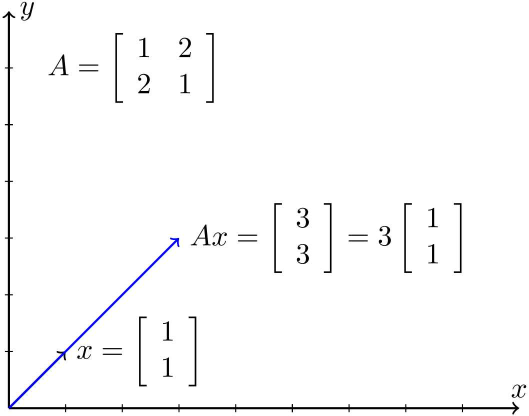
y
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
$Ax = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
x



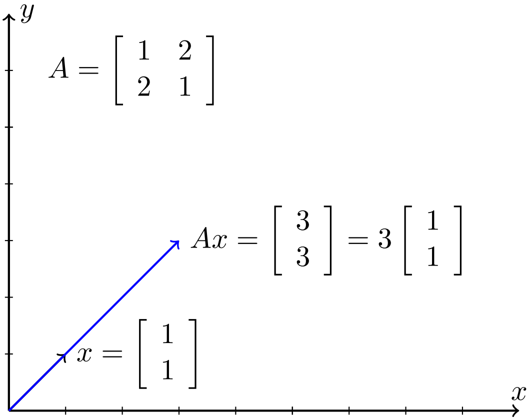
- So what is so special about eigenvectors?



- So what is so special about eigenvectors?
- Why are they always in the limelight?

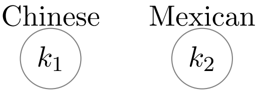


- So what is so special about eigenvectors?
- Why are they always in the limelight?
- It turns out that several properties of matrices can be analyzed based on their eigenvalues (for example, see spectral graph theory)



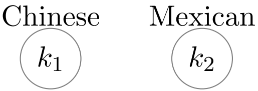
- So what is so special about eigenvectors?
- Why are they always in the limelight?
- It turns out that several properties of matrices can be analyzed based on their eigenvalues (for example, see spectral graph theory)
- We will now see two cases where eigenvalues/vectors will help us in this course

- Let us assume that on day 0, k_1 students eat Chinese food, and k_2 students eat Mexican food. (Of course, no one eats in the mess!)



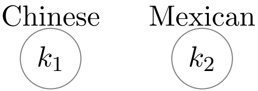
$$v_{(0)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

- Let us assume that on day 0, k_1 students eat Chinese food, and k_2 students eat Mexican food. (Of course, no one eats in the mess!)



$$v_{(0)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

- Let us assume that on day 0, k_1 students eat Chinese food, and k_2 students eat Mexican food. (Of course, no one eats in the mess!)
- On each subsequent day i, a fraction p of the students who ate Chinese food on day $(i-1)$, continue to eat Chinese food on day i, and $(1-p)$ shift to Mexican food.



$$v_{(0)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

- Let us assume that on day 0, k_1 students eat Chinese food, and k_2 students eat Mexican food. (Of course, no one eats in the mess!)
- On each subsequent day i, a fraction p of the students who ate Chinese food on day $(i-1)$, continue to eat Chinese food on day i, and $(1-p)$ shift to Mexican food.
- Similarly a fraction q of students who ate Mexican food on day $(i-1)$ continue to eat Mexican food on day i, and $(1-q)$ shift to Chinese food.

Chinese k_1 Mexican k_2

$$v_{(0)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$v_{(1)} = \begin{bmatrix} pk_1 + (1-q)k_2 \\ (1-p)k_1 + qk_2 \end{bmatrix}$$

- Let us assume that on day 0, k_1 students eat Chinese food, and k_2 students eat Mexican food. (Of course, no one eats in the mess!)
- On each subsequent day i, a fraction p of the students who ate Chinese food on day $(i-1)$, continue to eat Chinese food on day i, and $(1-p)$ shift to Mexican food.
- Similarly a fraction q of students who ate Mexican food on day $(i-1)$ continue to eat Mexican food on day i, and $(1-q)$ shift to Chinese food.

Chinese k_1 Mexican k_2

$$v_{(0)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$v_{(1)} = \begin{bmatrix} pk_1 + (1-q)k_2 \\ (1-p)k_1 + qk_2 \end{bmatrix}$$
$$= \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

- Let us assume that on day 0, k_1 students eat Chinese food, and k_2 students eat Mexican food. (Of course, no one eats in the mess!)
- On each subsequent day i, a fraction p of the students who ate Chinese food on day $(i-1)$, continue to eat Chinese food on day i, and $(1-p)$ shift to Mexican food.
- Similarly a fraction q of students who ate Mexican food on day $(i-1)$ continue to eat Mexican food on day i, and $(1-q)$ shift to Chinese food.

Chinese	Mexican
k_1	k_2

$$v_{(0)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\begin{aligned} v_{(1)} &= \begin{bmatrix} pk_1 + (1-q)k_2 \\ (1-p)k_1 + qk_2 \end{bmatrix} \\ &= \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} v_{(1)} &= Mv_{(0)} \\ v_{(2)} &= Mv_{(1)} \\ &= M^2 v_{(0)} \end{aligned}$$

- Let us assume that on day 0, k_1 students eat Chinese food, and k_2 students eat Mexican food. (Of course, no one eats in the mess!)
- On each subsequent day i, a fraction p of the students who ate Chinese food on day $(i-1)$, continue to eat Chinese food on day i, and $(1-p)$ shift to Mexican food.
- Similarly a fraction q of students who ate Mexican food on day $(i-1)$ continue to eat Mexican food on day i, and $(1-q)$ shift to Chinese food.

Chinese k_1 Mexican k_2

$$v_{(0)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

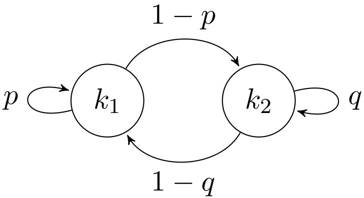
$$\begin{aligned} v_{(1)} &= \begin{bmatrix} pk_1 + (1-q)k_2 \\ (1-p)k_1 + qk_2 \end{bmatrix} \\ &= \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} v_{(1)} &= Mv_{(0)} \\ v_{(2)} &= Mv_{(1)} \\ &= M^2 v_{(0)} \end{aligned}$$

In general, $v_{(n)} = M^n v_{(0)}$

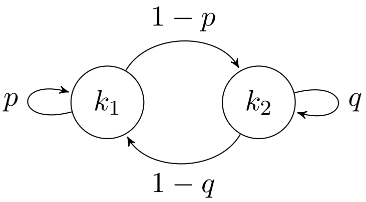
- Let us assume that on day 0, k_1 students eat Chinese food, and k_2 students eat Mexican food. (Of course, no one eats in the mess!)
- On each subsequent day i, a fraction p of the students who ate Chinese food on day $(i-1)$, continue to eat Chinese food on day i, and $(1-p)$ shift to Mexican food.
- Similarly a fraction q of students who ate Mexican food on day $(i-1)$ continue to eat Mexican food on day i, and $(1-q)$ shift to Chinese food.
- The number of customers in the two restaurants is thus given by the following series:

$$v_{(0)}, Mv_{(0)}, M^2 v_{(0)}, M^3 v_{(0)}, \ldots$$

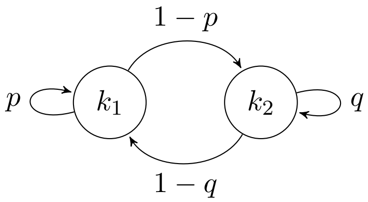


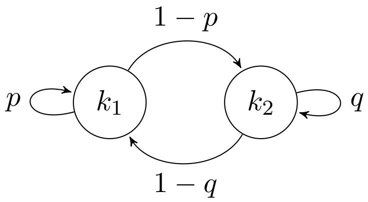
$1-p$
p
k_1
k_2
q
$1-q$

- This is a problem for the two restaurant owners.

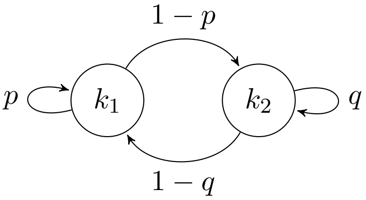


- This is a problem for the two restaurant owners.
- The number of patrons is changing constantly.

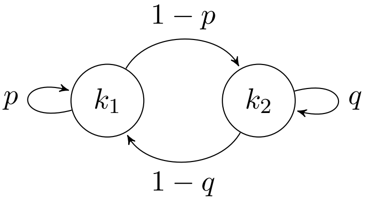




- This is a problem for the two restaurant owners.
- The number of patrons is changing constantly.
- Or is it? Will the system eventually reach a steady state? (i.e. will the number of customers in the two restaurants become constant over time?)



- This is a problem for the two restaurant owners.
- The number of patrons is changing constantly.
- Or is it? Will the system eventually reach a steady state? (i.e. will the number of customers in the two restaurants become constant over time?)
- Turns out they will!



- This is a problem for the two restaurant owners.
- The number of patrons is changing constantly.
- Or is it? Will the system eventually reach a steady state? (i.e. will the number of customers in the two restaurants become constant over time?)
- Turns out they will!
- Let's see how?

Definition

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvectors of an $n \times n$ matrix A. λ_1 is called the dominant eigen value of A if

$$|\lambda_1| \geq |\lambda_i| \; i = 2, \ldots, n$$

Definition

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvectors of an $n \times n$ matrix A. λ_1 is called the dominant eigen value of A if

$$|\lambda_1| \geq |\lambda_i| \ i = 2, \ldots, n$$

Definition

A matrix M is called a stochastic matrix if all the entries are positive and the sum of the elements in each column is equal to 1.
(Note that the matrix in our example is a stochastic matrix)

Definition

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvectors of an $n \times n$ matrix A. λ_1 is called the dominant eigen value of A if

$$|\lambda_1| \geq |\lambda_i| \; i = 2, \ldots, n$$

Definition

A matrix M is called a stochastic matrix if all the entries are positive and the sum of the elements in each column is equal to 1.
(Note that the matrix in our example is a stochastic matrix)

Theorem

The largest (dominant) eigenvalue of a stochastic matrix is 1.
See proof here

Definition

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvectors of an $n \times n$ matrix A. λ_1 is called the dominant eigen value of A if

$$|\lambda_1| \geq |\lambda_i| \; i = 2, \ldots, n$$

Theorem

The largest (dominant) eigenvalue of a stochastic matrix is 1.
See proof here

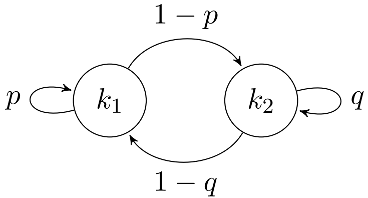
Definition

A matrix M is called a stochastic matrix if all the entries are positive and the sum of the elements in each column is equal to 1.
(Note that the matrix in our example is a stochastic matrix)

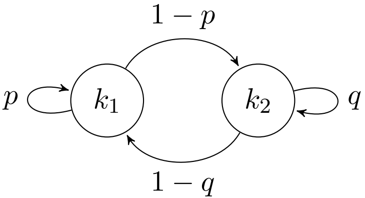
Theorem

If A is a $n \times n$ square matrix with a dominant eigenvalue, then the sequence of vectors given by $Av_0, A^2v_0, \ldots, A^nv_0, \ldots$ approaches a multiple of the dominant eigenvector of A.
(the theorem is slightly misstated here for ease of explanation)

- Let e_d be the dominant eigenvector of M and $\lambda_d = 1$ the corresponding dominant eigenvalue

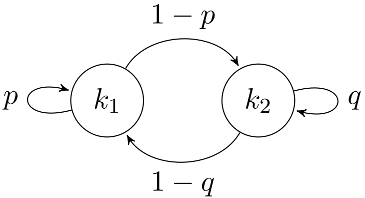


- Let e_d be the dominant eigenvector of M and $\lambda_d = 1$ the corresponding dominant eigenvalue
- Given the previous definitions and theorems, what can you say about the sequence $Mv_{(0)}, M^2v_{(0)}, M^3v_{(0)}, \ldots$?



- Let e_d be the dominant eigenvector of M and $\lambda_d = 1$ the corresponding dominant eigenvalue
- Given the previous definitions and theorems, what can you say about the sequence $Mv_{(0)}, M^2v_{(0)}, M^3v_{(0)}, \ldots$?
- There exists an n such that

$$v_{(n)} = M^n v_{(0)} = ke_d \text{ (some multiple of } e_d\text{)}$$

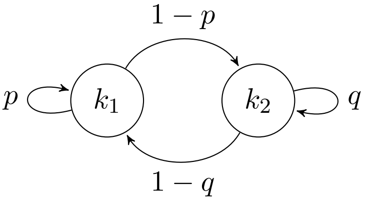


- Let e_d be the dominant eigenvector of M and $\lambda_d = 1$ the corresponding dominant eigenvalue
- Given the previous definitions and theorems, what can you say about the sequence $Mv_{(0)}, M^2v_{(0)}, M^3v_{(0)}, \ldots$?
- There exists an n such that

$$v_{(n)} = M^n v_{(0)} = ke_d \text{ (some multiple of } e_d)$$

- Now what happens at time step $(n+1)$?

$$v_{(n+1)} = Mv_{(n)} = M(ke_d) = k(Me_d) = k(\lambda_d e_d) = ke_d$$



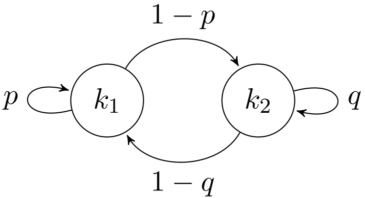
- Let e_d be the dominant eigenvector of M and $\lambda_d = 1$ the corresponding dominant eigenvalue
- Given the previous definitions and theorems, what can you say about the sequence $Mv_{(0)}, M^2v_{(0)}, M^3v_{(0)}, \ldots$?
- There exists an n such that

$$v_{(n)} = M^n v_{(0)} = ke_d \text{ (some multiple of } e_d\text{)}$$

- Now what happens at time step $(n+1)$?

$$v_{(n+1)} = Mv_{(n)} = M(ke_d) = k(Me_d) = k(\lambda_d e_d) = ke_d$$

- The population in the two restaurants becomes constant after time step n.
  See Proof Here



- Now instead of a stochastic matrix let us consider any square matrix A

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$A^p x_0 = ke_d$$

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$A^p x_0 = k e_d$$
$$A^{p+1} x_0 = A(A^p x_0) = kAe_d = k\lambda_d e_d$$

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$
\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d
\end{aligned}
$$

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$
\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d \\
A^{p+n} x_0 &=
\end{aligned}
$$

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$
\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d \\
A^{p+n} x_0 &= k(\lambda_d)^n e_d
\end{aligned}
$$

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$
\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d \\
A^{p+n} x_0 &= k(\lambda_d)^n e_d
\end{aligned}
$$

- In general, if λ_d is the dominant eigenvalue of a matrix A, what would happen to the sequence $x_0, Ax_0, A^2x_0, \ldots$ if

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \dots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$
\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d \\
A^{p+n} x_0 &= k(\lambda_d)^n e_d
\end{aligned}
$$

- In general, if λ_d is the dominant eigenvalue of a matrix A, what would happen to the sequence $x_0, Ax_0, A^2x_0, \dots$ if
  - $|\lambda_d| > 1$

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d \\
A^{p+n} x_0 &= k(\lambda_d)^n e_d
\end{aligned}$$

- In general, if λ_d is the dominant eigenvalue of a matrix A, what would happen to the sequence $x_0, Ax_0, A^2x_0, \ldots$ if
  - $|\lambda_d| > 1$ (will explode)

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d \\
A^{p+n} x_0 &= k(\lambda_d)^n e_d
\end{aligned}$$

- In general, if λ_d is the dominant eigenvalue of a matrix A, what would happen to the sequence $x_0, Ax_0, A^2x_0, \ldots$ if
  - $|\lambda_d| > 1$ (will explode)
  - $|\lambda_d| < 1$

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$
\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d \\
A^{p+n} x_0 &= k(\lambda_d)^n e_d
\end{aligned}
$$

- In general, if λ_d is the dominant eigenvalue of a matrix A, what would happen to the sequence $x_0, Ax_0, A^2x_0, \ldots$ if
  - $|\lambda_d| > 1$ (will explode)
  - $|\lambda_d| < 1$ (will vanish)

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d \\
A^{p+n} x_0 &= k(\lambda_d)^n e_d
\end{aligned}$$

- In general, if λ_d is the dominant eigenvalue of a matrix A, what would happen to the sequence $x_0, Ax_0, A^2x_0, \ldots$ if
  - $|\lambda_d| > 1$ (will explode)
  - $|\lambda_d| < 1$ (will vanish)
  - $|\lambda_d| = 1$

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$
\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d \\
A^{p+n} x_0 &= k(\lambda_d)^n e_d
\end{aligned}
$$

- In general, if λ_d is the dominant eigenvalue of a matrix A, what would happen to the sequence $x_0, Ax_0, A^2x_0, \ldots$ if
  - $|\lambda_d| > 1$ (will explode)
  - $|\lambda_d| < 1$ (will vanish)
  - $|\lambda_d| = 1$ (will reach a steady state)

- Now instead of a stochastic matrix let us consider any square matrix A
- Let p be the time step at which the sequence $x_0, Ax_0, A^2x_0, \ldots$ approaches a multiple of e_d (the dominant eigenvector of A)

$$
\begin{aligned}
A^p x_0 &= k e_d \\
A^{p+1} x_0 &= A(A^p x_0) = kAe_d = k\lambda_d e_d \\
A^{p+2} x_0 &= A(A^{p+1} x_0) = k\lambda_d A e_d = k\lambda_d^2 e_d \\
A^{p+n} x_0 &= k(\lambda_d)^n e_d
\end{aligned}
$$

- In general, if λ_d is the dominant eigenvalue of a matrix A, what would happen to the sequence $x_0, Ax_0, A^2x_0, \ldots$ if
  - $|\lambda_d| > 1$ (will explode)
  - $|\lambda_d| < 1$ (will vanish)
  - $|\lambda_d| = 1$ (will reach a steady state)
- (We will use this in the course at some point)

Module 6.2 : Linear Algebra - Basic Definitions

- We will see some more examples where eigenvectors are important, but before that let's revisit some basic definitions from linear algebra.

Basis

A set of vectors $\in \mathbb{R}^n$ is called a basis, if they are linearly independent and every vector $\in \mathbb{R}^n$ can be expressed as a linear combination of these vectors.

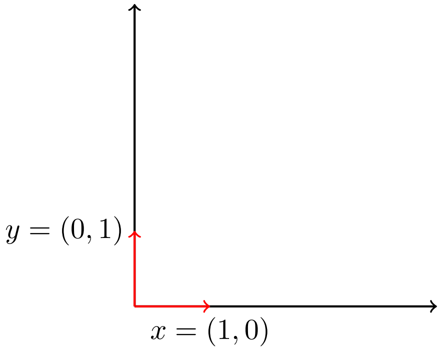
Basis

A set of vectors $\in \mathbb{R}^n$ is called a basis, if they are linearly independent and every vector $\in \mathbb{R}^n$ can be expressed as a linear combination of these vectors.

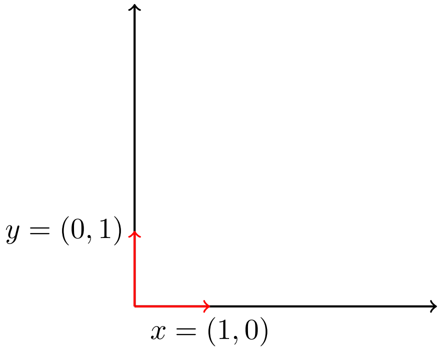
Linearly independent vectors

A set of n vectors $v_1, v_2, \ldots, v_n$ is linearly independent if no vector in the set can be expressed as a linear combination of the remaining $n-1$ vectors.
In other words, the only solution to

$$c_1 v_1 + c_2 v_2 + \ldots c_n v_n = 0 \text{ is } c_1 = c_2 = \cdots = c_n = 0 (c_i\text{'s } \textit{are scalars})$$

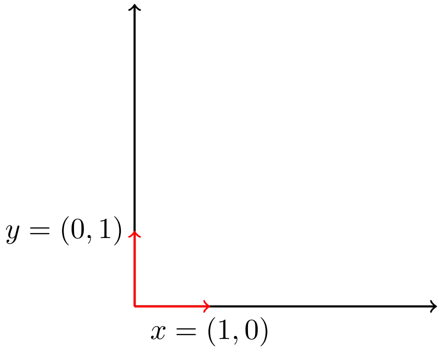


- For example consider the space $\mathbb{R}^2$



- For example consider the space $\mathbb{R}^2$
- Now consider the vectors

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

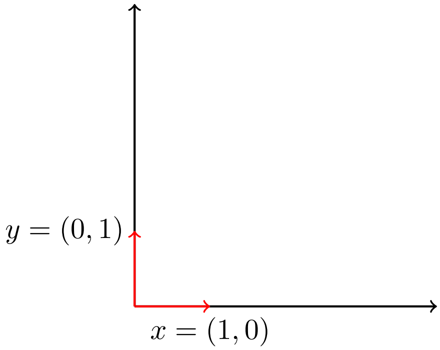


- For example consider the space $\mathbb{R}^2$
- Now consider the vectors

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Any vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, can be expressed as a linear combination of these two vectors i.e

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



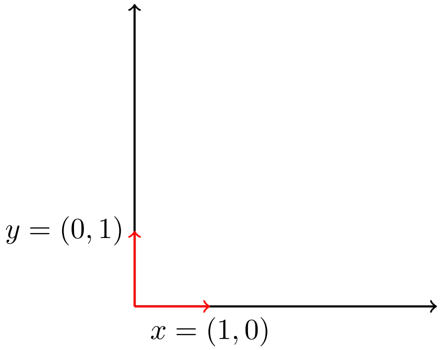
- For example consider the space $\mathbb{R}^2$
- Now consider the vectors

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

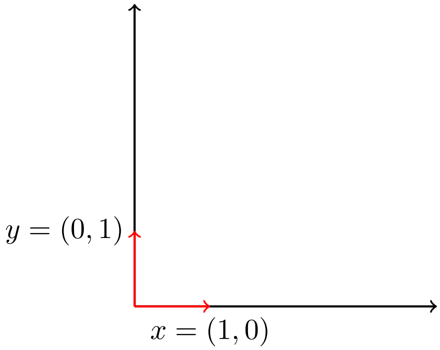
- Any vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, can be expressed as a linear combination of these two vectors i.e

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

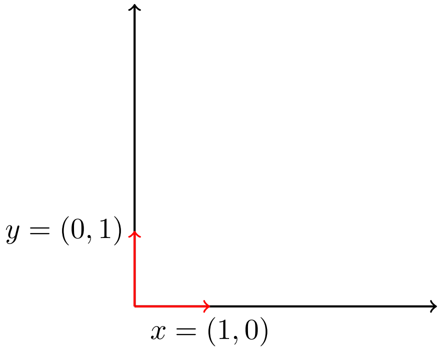
- Further, x and y are linearly independent. (the only solution to $c_1 x + c_2 y = 0$ is $c_1 = c_2 = 0$)



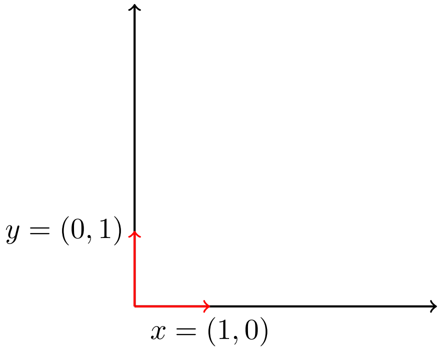
- In fact, turns out that x and y are unit vectors in the direction of the co-ordinate axes.



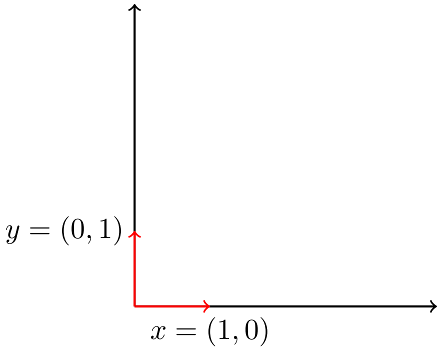
- In fact, turns out that x and y are unit vectors in the direction of the co-ordinate axes.
- And indeed we are used to representing all vectors in $\mathbb{R}^2$ as a linear combination of these two vectors.



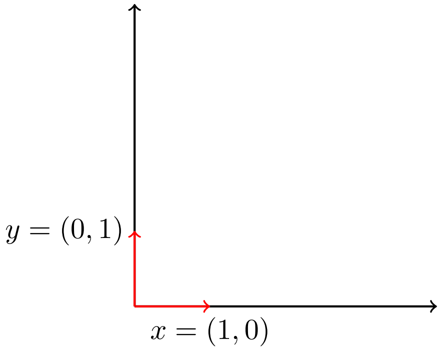
- In fact, turns out that x and y are unit vectors in the direction of the co-ordinate axes.
- And indeed we are used to representing all vectors in $\mathbb{R}^2$ as a linear combination of these two vectors.
- But there is nothing sacrosanct about the particular choice of x and y.



- In fact, turns out that x and y are unit vectors in the direction of the co-ordinate axes.
- And indeed we are used to representing all vectors in $\mathbb{R}^2$ as a linear combination of these two vectors.
- But there is nothing sacrosanct about the particular choice of x and y.
- We could have chosen any 2 linearly independent vectors in $\mathbb{R}^2$ as the basis vectors.

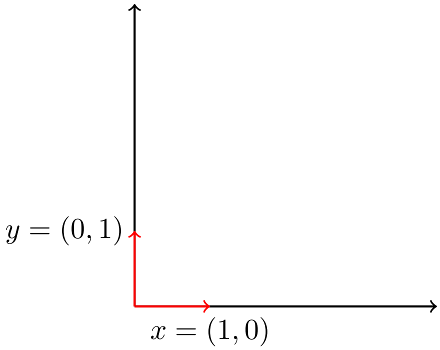


- In fact, turns out that x and y are unit vectors in the direction of the co-ordinate axes.
- And indeed we are used to representing all vectors in $\mathbb{R}^2$ as a linear combination of these two vectors.
- But there is nothing sacrosanct about the particular choice of x and y.
- We could have chosen any 2 linearly independent vectors in $\mathbb{R}^2$ as the basis vectors.
- For example, consider the linearly independent vectors, $[2, 3]^T$ and $[5, 7]^T$. See how any vector $[a, b]^T \in \mathbb{R}^2$ can be expressed as a linear combination of these two vectors.



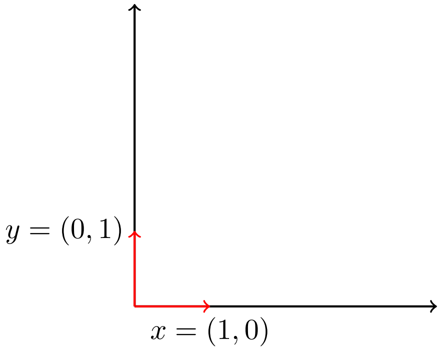
$$\begin{bmatrix} a \\ b \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

- In fact, turns out that x and y are unit vectors in the direction of the co-ordinate axes.
- And indeed we are used to representing all vectors in $\mathbb{R}^2$ as a linear combination of these two vectors.
- But there is nothing sacrosanct about the particular choice of x and y.
- We could have chosen any 2 linearly independent vectors in $\mathbb{R}^2$ as the basis vectors.
- For example, consider the linearly independent vectors, $[2, 3]^T$ and $[5, 7]^T$. See how any vector $[a, b]^T \in \mathbb{R}^2$ can be expressed as a linear combination of these two vectors.



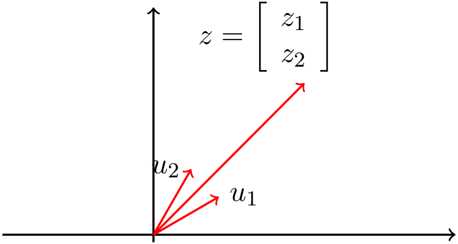
$$\begin{bmatrix} a \\ b \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

- In fact, turns out that x and y are unit vectors in the direction of the co-ordinate axes.
- And indeed we are used to representing all vectors in $\mathbb{R}^2$ as a linear combination of these two vectors.
- But there is nothing sacrosanct about the particular choice of x and y.
- We could have chosen any 2 linearly independent vectors in $\mathbb{R}^2$ as the basis vectors.
- For example, consider the linearly independent vectors, $[2,3]^T$ and $[5,7]^T$. See how any vector $[a,b]^T \in \mathbb{R}^2$ can be expressed as a linear combination of these two vectors.
- We can find x_1 and x_2 by solving a system of linear equations.

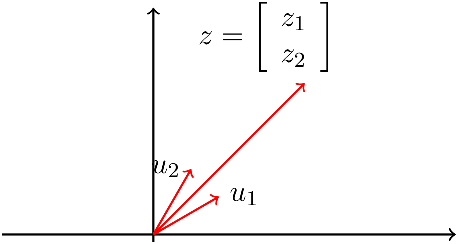


$$a = 2x_1 + 5x_2$$
$$b = 3x_1 + 7x_2$$

- In fact, turns out that x and y are unit vectors in the direction of the co-ordinate axes.
- And indeed we are used to representing all vectors in $\mathbb{R}^2$ as a linear combination of these two vectors.
- But there is nothing sacrosanct about the particular choice of x and y.
- We could have chosen any 2 linearly independent vectors in $\mathbb{R}^2$ as the basis vectors.
- For example, consider the linearly independent vectors, $[2,3]^T$ and $[5,7]^T$. See how any vector $[a,b]^T \in \mathbb{R}^2$ can be expressed as a linear combination of these two vectors.
- We can find x_1 and x_2 by solving a system of linear equations.

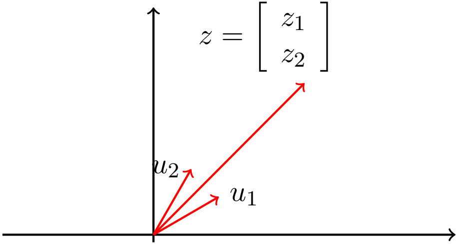


- In general, given a set of linearly independent vectors $u_1, u_2, \ldots u_n \in \mathbb{R}^n$, we can express any vector $z \in \mathbb{R}^n$ as a linear combination of these vectors.



- In general, given a set of linearly independent vectors $u_1, u_2, \ldots u_n \in \mathbb{R}^n$, we can express any vector $z \in \mathbb{R}^n$ as a linear combination of these vectors.

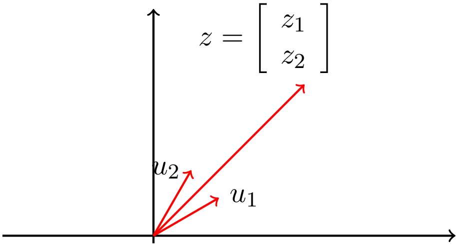
$$z = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$



- In general, given a set of linearly independent vectors $u_1, u_2, \ldots u_n \in \mathbb{R}^n$, we can express any vector $z \in \mathbb{R}^n$ as a linear combination of these vectors.

$$z = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \alpha_1 \begin{bmatrix} u_{11} \\ u_{12} \\ \vdots \\ u_{1n} \end{bmatrix} + \alpha_2 \begin{bmatrix} u_{21} \\ u_{22} \\ \vdots \\ u_{2n} \end{bmatrix} + \ldots + \alpha_n \begin{bmatrix} u_{n1} \\ u_{n2} \\ \vdots \\ u_{nn} \end{bmatrix}$$



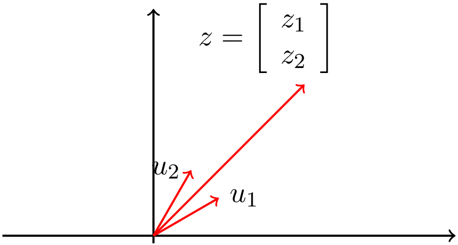
- In general, given a set of linearly independent vectors $u_1, u_2, \ldots u_n \in \mathbb{R}^n$, we can express any vector $z \in \mathbb{R}^n$ as a linear combination of these vectors.

$$z = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \alpha_1 \begin{bmatrix} u_{11} \\ u_{12} \\ \vdots \\ u_{1n} \end{bmatrix} + \alpha_2 \begin{bmatrix} u_{21} \\ u_{22} \\ \vdots \\ u_{2n} \end{bmatrix} + \ldots + \alpha_n \begin{bmatrix} u_{n1} \\ u_{n2} \\ \vdots \\ u_{nn} \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} u_{11} & u_{21} & \ldots & u_{n1} \\ u_{12} & u_{22} & \ldots & u_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ u_{1n} & u_{2n} & \ldots & u_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

(Basically rewriting in matrix form)



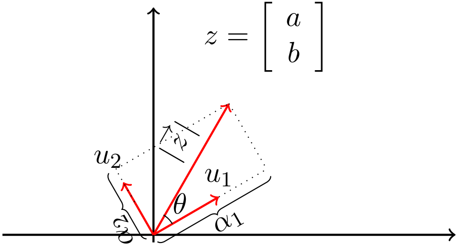
- In general, given a set of linearly independent vectors $u_1, u_2, \ldots u_n \in \mathbb{R}^n$, we can express any vector $z \in \mathbb{R}^n$ as a linear combination of these vectors.

$$z = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

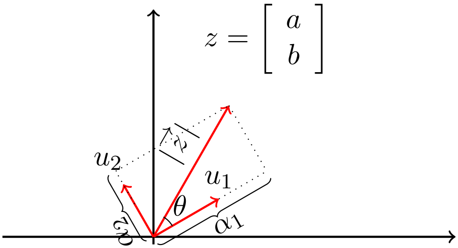
$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \alpha_1 \begin{bmatrix} u_{11} \\ u_{12} \\ \vdots \\ u_{1n} \end{bmatrix} + \alpha_2 \begin{bmatrix} u_{21} \\ u_{22} \\ \vdots \\ u_{2n} \end{bmatrix} + \ldots + \alpha_n \begin{bmatrix} u_{n1} \\ u_{n2} \\ \vdots \\ u_{nn} \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} u_{11} & u_{21} & \ldots & u_{n1} \\ u_{12} & u_{22} & \ldots & u_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ u_{1n} & u_{2n} & \ldots & u_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

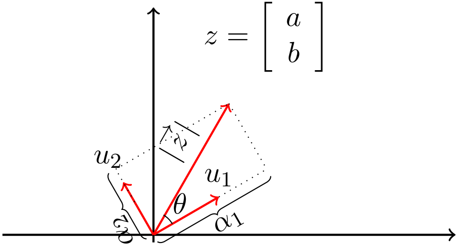
- We can now find the α_is using Gaussian Elimination (Time Complexity: $O(n^3)$)



- Now let us see if we have orthonormal basis.

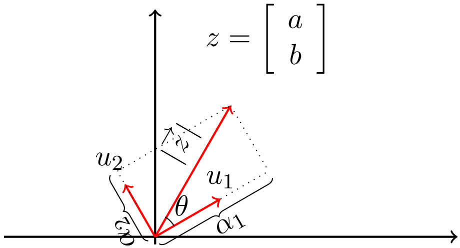


- Now let us see if we have orthonormal basis.
- $u_i^T u_j = 0 \; \forall i \neq j$ and $u_i^T u_i = \|u_i\|^2 = 1$



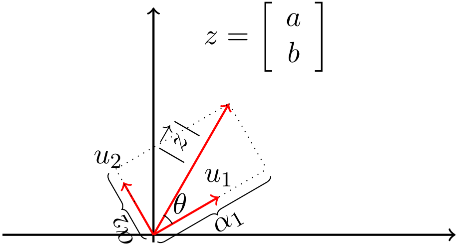
- Now let us see if we have orthonormal basis.
- $u_i^T u_j = 0 \ \forall i \neq j$ and $u_i^T u_i = \|u_i\|^2 = 1$
- Again we have:

$$z = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$$



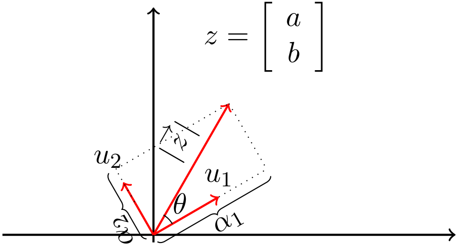
- Now let us see if we have orthonormal basis.
- $u_i^T u_j = 0 \ \forall i \neq j$ and $u_i^T u_i = \|u_i\|^2 = 1$
- Again we have:

$$z = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$$
$$u_1^T z = \alpha_1 u_1^T u_1 + \ldots + \alpha_n u_1^T u_n$$



- Now let us see if we have orthonormal basis.
- $u_i^T u_j = 0 \ \forall i \neq j$ and $u_i^T u_i = \|u_i\|^2 = 1$
- Again we have:

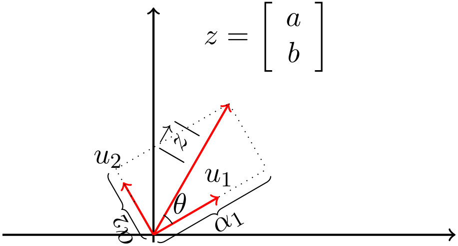
$$
\begin{aligned}
z &= \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \\
u_1^T z &= \alpha_1 u_1^T u_1 + \ldots + \alpha_n u_1^T u_n \\
&= \alpha_1
\end{aligned}
$$



- Now let us see if we have orthonormal basis.
- $u_i^T u_j = 0 \ \forall i \neq j$ and $u_i^T u_i = \|u_i\|^2 = 1$
- Again we have:

$$
\begin{aligned}
z &= \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \\
u_1^T z &= \alpha_1 u_1^T u_1 + \ldots + \alpha_n u_1^T u_n \\
&= \alpha_1
\end{aligned}
$$

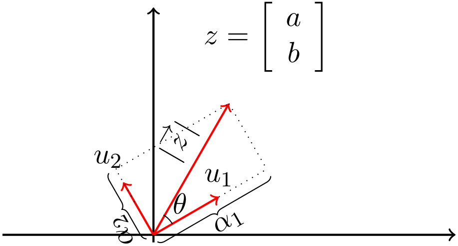
- We can directly find each α_i using a dot product between z and u_i (time complexity $O(N)$)



- Now let us see if we have orthonormal basis.
- $u_i^T u_j = 0 \ \forall i \neq j$ and $u_i^T u_i = \|u_i\|^2 = 1$
- Again we have:

$$
\begin{aligned}
z &= \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \\
u_1^T z &= \alpha_1 u_1^T u_1 + \ldots + \alpha_n u_1^T u_n \\
&= \alpha_1
\end{aligned}
$$

- We can directly find each α_i using a dot product between z and u_i (time complexity $O(N)$)
- The total complexity will be $O(N^2)$

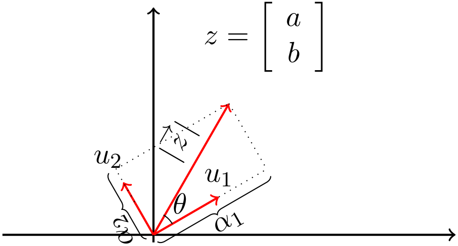


$$\alpha_1 = |\vec{z}| cos\theta = |\vec{z}| \frac{z^T u_1}{|\vec{z}||u_1|} = z^T u_1$$

- Now let us see if we have orthonormal basis.
- $u_i^T u_j = 0 \; \forall i \neq j$ and $u_i^T u_i = \|u_i\|^2 = 1$
- Again we have:

$$\begin{aligned} z &= \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \\ u_1^T z &= \alpha_1 u_1^T u_1 + \ldots + \alpha_n u_1^T u_n \\ &= \alpha_1 \end{aligned}$$

- We can directly find each α_i using a dot product between z and u_i (time complexity $O(N)$)
- The total complexity will be $O(N^2)$



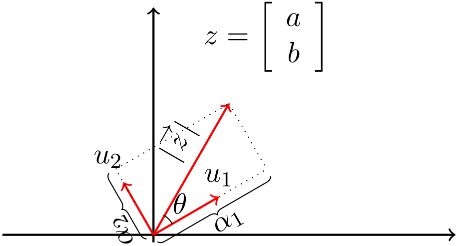
$$\alpha_1 = |\vec{z}|cos\theta = |\vec{z}|\frac{z^T u_1}{|\vec{z}||u_1|} = z^T u_1$$

Similarly, $\alpha_2 = z^T u_2$.

- Now let us see if we have orthonormal basis.
- $u_i^T u_j = 0 \ \forall i \neq j$ and $u_i^T u_i = \|u_i\|^2 = 1$
- Again we have:

$$\begin{aligned} z &= \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \\ u_1^T z &= \alpha_1 u_1^T u_1 + \ldots + \alpha_n u_1^T u_n \\ &= \alpha_1 \end{aligned}$$

- We can directly find each α_i using a dot product between z and u_i (time complexity $O(N)$)
- The total complexity will be $O(N^2)$



$$\alpha_1 = |\vec{z}|cos\theta = |\vec{z}|\frac{z^T u_1}{|\vec{z}||u_1|} = z^T u_1$$

Similarly, $\alpha_2 = z^T u_2$.
When u_1 and u_2 are unit vectors along the co-ordinate axes

$$z = \begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Now let us see if we have orthonormal basis.
- $u_i^T u_j = 0 \ \forall i \neq j$ and $u_i^T u_i = \|u_i\|^2 = 1$
- Again we have:

$$\begin{aligned} z &= \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \\ u_1^T z &= \alpha_1 u_1^T u_1 + \ldots + \alpha_n u_1^T u_n \\ &= \alpha_1 \end{aligned}$$

- We can directly find each α_i using a dot product between z and u_i (time complexity $O(N)$)
- The total complexity will be $O(N^2)$

Remember

An orthogonal basis is the most convenient basis that one can hope for.

- But what does any of this have to do with eigenvectors?

- But what does any of this have to do with eigenvectors?
- Turns out that the eigenvectors can form a basis.

Theorem 1

The eigenvectors of a matrix $A \in \mathbb{R}^{n \times n}$ having distinct eigenvalues are linearly independent.

Proof: See here

- But what does any of this have to do with eigenvectors?
- Turns out that the eigenvectors can form a basis.

Theorem 1

The eigenvectors of a matrix $A \in \mathbb{R}^{n \times n}$ having distinct eigenvalues are linearly independent.

Proof: See here

- But what does any of this have to do with eigenvectors?
- Turns out that the eigenvectors can form a basis.
- In fact, the eigenvectors of a square symmetric matrix are even more special.

Theorem 1

The eigenvectors of a matrix $A \in \mathbb{R}^{n \times n}$ having distinct eigenvalues are linearly independent.

Proof: See here

Theorem 2

The eigenvectors of a square symmetric matrix are orthogonal.

Proof: See here

- But what does any of this have to do with eigenvectors?
- Turns out that the eigenvectors can form a basis.
- In fact, the eigenvectors of a square symmetric matrix are even more special.

Theorem 1

The eigenvectors of a matrix $A \in \mathbb{R}^{n \times n}$ having distinct eigenvalues are linearly independent.

Proof: See here

Theorem 2

The eigenvectors of a square symmetric matrix are orthogonal.

Proof: See here

- But what does any of this have to do with eigenvectors?
- Turns out that the eigenvectors can form a basis.
- In fact, the eigenvectors of a square symmetric matrix are even more special.
- Thus they form a very convenient basis.

Theorem 1

The eigenvectors of a matrix $A \in \mathbb{R}^{n \times n}$ having distinct eigenvalues are linearly independent.

Proof: See here

Theorem 2

The eigenvectors of a square symmetric matrix are orthogonal.

Proof: See here

- But what does any of this have to do with eigenvectors?
- Turns out that the eigenvectors can form a basis.
- In fact, the eigenvectors of a square symmetric matrix are even more special.
- Thus they form a very convenient basis.
- Why would we want to use the eigenvectors as a basis instead of the more natural co-ordinate axes?

Theorem 1

The eigenvectors of a matrix $A \in \mathbb{R}^{n \times n}$ having distinct eigenvalues are linearly independent.

Proof: See here

Theorem 2

The eigenvectors of a square symmetric matrix are orthogonal.

Proof: See here

- But what does any of this have to do with eigenvectors?
- Turns out that the eigenvectors can form a basis.
- In fact, the eigenvectors of a square symmetric matrix are even more special.
- Thus they form a very convenient basis.
- Why would we want to use the eigenvectors as a basis instead of the more natural co-ordinate axes?
- We will answer this question soon.

Module 6.3 : Eigenvalue Decomposition

Before proceeding let's do a quick recap of eigenvalue decomposition.

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
- Consider a matrix U whose columns are $u_1, u_2, \ldots, u_n$.

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
- Consider a matrix U whose columns are $u_1, u_2, \ldots, u_n$.
- Now

$$AU =$$

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
- Consider a matrix U whose columns are $u_1, u_2, \ldots, u_n$.
- Now

$$AU = A \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
- Consider a matrix U whose columns are $u_1, u_2, \ldots, u_n$.
- Now

$$AU = A \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ Au_1 & Au_2 & \ldots & Au_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
- Consider a matrix U whose columns are $u_1, u_2, \ldots, u_n$.
- Now

$$AU = A \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ Au_1 & Au_2 & \ldots & Au_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \lambda_1 u_1 & \lambda_2 u_2 & \ldots & \lambda_n u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
- Consider a matrix U whose columns are $u_1, u_2, \ldots, u_n$.
- Now

$$AU = A\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ Au_1 & Au_2 & \ldots & Au_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \lambda_1 u_1 & \lambda_2 u_2 & \ldots & \lambda_n u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
- Consider a matrix U whose columns are $u_1, u_2, \ldots, u_n$.
- Now

$$AU = A\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ Au_1 & Au_2 & \ldots & Au_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \lambda_1 u_1 & \lambda_2 u_2 & \ldots & \lambda_n u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix}$$

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
- Consider a matrix U whose columns are $u_1, u_2, \ldots, u_n$.
- Now

$$AU = A \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ Au_1 & Au_2 & \ldots & Au_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \lambda_1 u_1 & \lambda_2 u_2 & \ldots & \lambda_n u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix} = U\Lambda$$

- Let $u_1, u_2, \ldots, u_n$ be the eigenvectors of a matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
- Consider a matrix U whose columns are $u_1, u_2, \ldots, u_n$.
- Now

$$
\begin{aligned}
AU = A\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} &= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ Au_1 & Au_2 & \ldots & Au_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \\
&= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \lambda_1 u_1 & \lambda_2 u_2 & \ldots & \lambda_n u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \\
&= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \ldots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix} = U\Lambda
\end{aligned}
$$

- where Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A.

$$AU = U\Lambda$$

$$AU = U\Lambda$$

- If U^{-1} exists, then we can write,

$$A = U\Lambda U^{-1} \quad \text{[eigenvalue decomposition]}$$
$$U^{-1}AU = \Lambda \quad \text{[diagonalization of A]}$$

$$AU = U\Lambda$$

- If U^{-1} exists, then we can write,

$$A = U\Lambda U^{-1} \qquad \text{[eigenvalue decomposition]}$$
$$U^{-1}AU = \Lambda \qquad \text{[diagonalization of A]}$$

- Under what conditions would U^{-1} exist?

$$AU = U\Lambda$$

- If U^{-1} exists, then we can write,

$$A = U\Lambda U^{-1} \qquad \text{[eigenvalue decomposition]}$$
$$U^{-1}AU = \Lambda \qquad \text{[diagonalization of A]}$$

- Under what conditions would U^{-1} exist?
  - If the columns of U are linearly independent [**See proof here**]

$$AU = U\Lambda$$

- If U^{-1} exists, then we can write,

$$A = U\Lambda U^{-1} \quad \text{[eigenvalue decomposition]}$$
$$U^{-1}AU = \Lambda \quad \text{[diagonalization of A]}$$

- Under what conditions would U^{-1} exist?
  - If the columns of U are linearly independent [**See proof here**]
  - *i.e.* if A has n linearly independent eigenvectors.

$$AU = U\Lambda$$

- If U^{-1} exists, then we can write,

$$A = U\Lambda U^{-1} \qquad \text{[eigenvalue decomposition]}$$
$$U^{-1}AU = \Lambda \qquad \text{[diagonalization of A]}$$

- Under what conditions would U^{-1} exist?
  - If the columns of U are linearly independent [**See proof here**]
  - *i.e.* if A has n linearly independent eigenvectors.
  - *i.e.* if A has n distinct eigenvalues [**sufficient condition, proof : Slide 19 Theorem 1**]

- If A is symmetric then the situation is even more convenient.

- If A is symmetric then the situation is even more convenient.
- The eigenvectors are orthogonal [**proof : Slide 19 Theorem 2**]

- If A is symmetric then the situation is even more convenient.
- The eigenvectors are orthogonal [**proof : Slide 19 Theorem 2**]
- Further let's assume, that the eigenvectors have been normalized [$u_i^T u_i = 1$]

$$Q = U^T U = \begin{bmatrix} \leftarrow u_1 \rightarrow \\ \leftarrow u_2 \rightarrow \\ \dots \\ \leftarrow u_n \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \dots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

- If A is symmetric then the situation is even more convenient.
- The eigenvectors are orthogonal [**proof : Slide 19 Theorem 2**]
- Further let's assume, that the eigenvectors have been normalized [$u_i^T u_i = 1$]

$$Q = U^T U = \begin{bmatrix} \leftarrow u_1 \rightarrow \\ \leftarrow u_2 \rightarrow \\ \dots \\ \leftarrow u_n \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \dots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

- Each cell of the matrix, Q_{ij} is given by $u_i^T u_j$

$$\begin{aligned} Q_{ij} = u_i^T u_j \quad & = 0 \text{ if } i \neq j \\ & = 1 \text{ if } i = j \end{aligned}$$

$$\therefore U^T U = \mathbb{I} \text{ (the identity matrix)}$$

- If A is symmetric then the situation is even more convenient.
- The eigenvectors are orthogonal [**proof : Slide 19 Theorem 2**]
- Further let's assume, that the eigenvectors have been normalized [$u_i^T u_i = 1$]

$$Q = U^T U = \begin{bmatrix} \leftarrow u_1 \rightarrow \\ \leftarrow u_2 \rightarrow \\ \dots \\ \leftarrow u_n \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \dots & u_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

- Each cell of the matrix, Q_{ij} is given by $u_i^T u_j$

$$\begin{aligned} Q_{ij} = u_i^T u_j \quad &= 0 \text{ if } i \neq j \\ &= 1 \text{ if } i = j \end{aligned}$$

$$\therefore U^T U = \mathbb{I} \text{ (the identity matrix)}$$

- U^T is the inverse of U (very convenient to calculate)

Something to think about

- Given the EVD, $A = U\Sigma U^T$,
  what can you say about the sequence $x_0, Ax_0, A^2x_0, \ldots$ in terms of the eigen values of A.
  (Hint: You should arrive at the same conclusion we saw earlier)

Theorem (one more important property of eigenvectors)

If A is a square symmetric $N \times N$ matrix, then the solution to the following optimization problem is given by the eigenvector corresponding to the largest eigenvalue of A.

$$\max_{x} \; x^T A x$$
$$\text{s.t } \|x\| = 1$$

and the solution to

$$\min_{x} \; x^T A x$$
$$\text{s.t } \|x\| = 1$$

is given by the eigenvector corresponding to the smallest eigenvalue of A.

Proof: Next slide.

- This is a constrained optimization problem that can be solved using Lagrange Multipliers:

$$L = x^T Ax - \lambda(x^T x - 1)$$

$$\frac{\partial L}{\partial x} = 2Ax - \lambda(2x) = 0 => Ax = \lambda x$$

- This is a constrained optimization problem that can be solved using Lagrange Multipliers:

$$L = x^T Ax - \lambda(x^T x - 1)$$

$$\frac{\partial L}{\partial x} = 2Ax - \lambda(2x) = 0 => Ax = \lambda x$$

- Hence x must be an eigenvector of A with eigenvalue λ.

- This is a constrained optimization problem that can be solved using Lagrange Multipliers:

$$L = x^T Ax - \lambda(x^T x - 1)$$
$$\frac{\partial L}{\partial x} = 2Ax - \lambda(2x) = 0 => Ax = \lambda x$$

- Hence x must be an eigenvector of A with eigenvalue λ.
- Multiplying by x^T:

$$x^T Ax = \lambda x^T x = \lambda (\text{since } x^T x = 1)$$

- This is a constrained optimization problem that can be solved using Lagrange Multipliers:

$$L = x^T A x - \lambda(x^T x - 1)$$
$$\frac{\partial L}{\partial x} = 2Ax - \lambda(2x) = 0 => Ax = \lambda x$$

- Hence x must be an eigenvector of A with eigenvalue λ.
- Multiplying by x^T:

$$x^T A x = \lambda x^T x = \lambda (\text{since } x^T x = 1)$$

- Therefore, the critical points of this constrained problem are the eigenvalues of A.

- This is a constrained optimization problem that can be solved using Lagrange Multipliers:

$$L = x^T Ax - \lambda(x^T x - 1)$$

$$\frac{\partial L}{\partial x} = 2Ax - \lambda(2x) = 0 => Ax = \lambda x$$

- Hence x must be an eigenvector of A with eigenvalue λ.
- Multiplying by x^T:

$$x^T Ax = \lambda x^T x = \lambda (\text{since } x^T x = 1)$$

- Therefore, the critical points of this constrained problem are the eigenvalues of A.
- The maximum value is the largest eigenvalue, while the minimum value is the smallest eigenvalue.

The story so far...

The story so far...

- The eigenvectors corresponding to different eigenvalues are linearly independent.

The story so far...

- The eigenvectors corresponding to different eigenvalues are linearly independent.
- The eigenvectors of a square symmetric matrix are orthogonal.

The story so far...

- The eigenvectors corresponding to different eigenvalues are linearly independent.
- The eigenvectors of a square symmetric matrix are orthogonal.
- The eigenvectors of a square symmetric matrix can thus form a convenient basis.

The story so far...

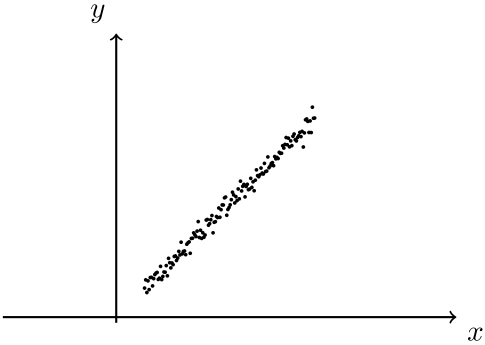
- The eigenvectors corresponding to different eigenvalues are linearly independent.
- The eigenvectors of a square symmetric matrix are orthogonal.
- The eigenvectors of a square symmetric matrix can thus form a convenient basis.
- We will put all of this to use.

Module 6.4 : Principal Component Analysis and its Interpretations

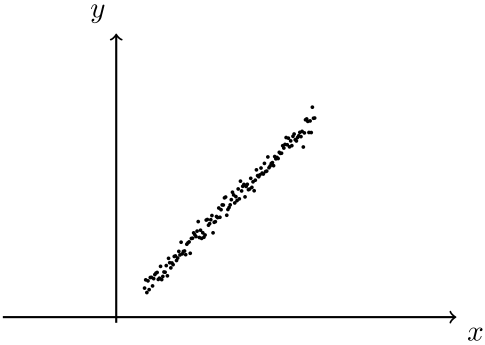
The story ahead...

The story ahead...

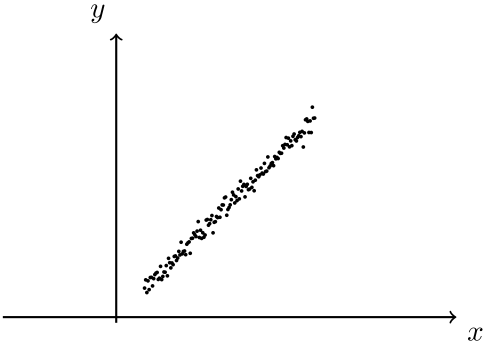
- Over the next few slides we will introduce Principal Component Analysis and see three different interpretations of it



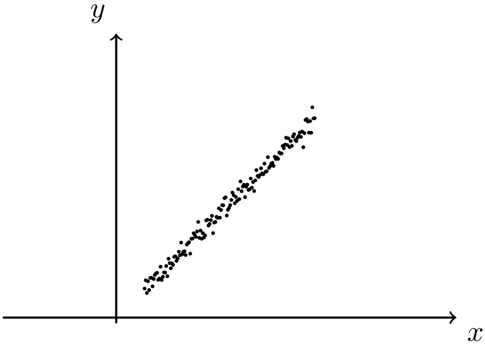
- Consider the following data



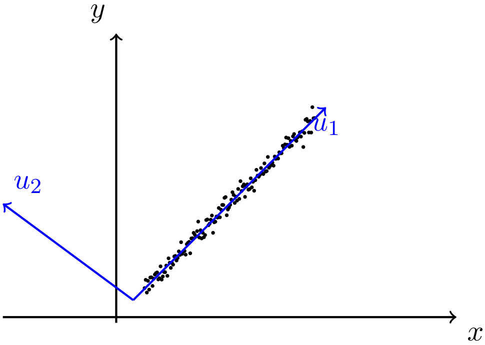
- Consider the following data
- Each point (vector) here is represented using a linear combination of the x and y axes (i.e. using the point's x and y co-ordinates)



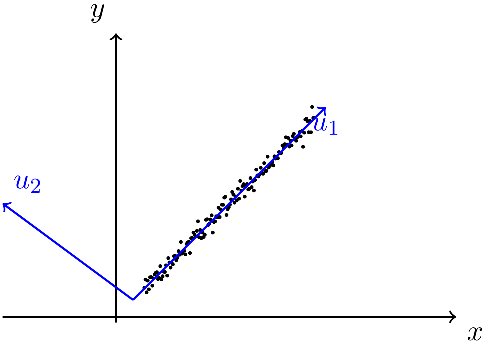
- Consider the following data
- Each point (vector) here is represented using a linear combination of the x and y axes (i.e. using the point's x and y co-ordinates)
- In other words we are using x and y as the basis



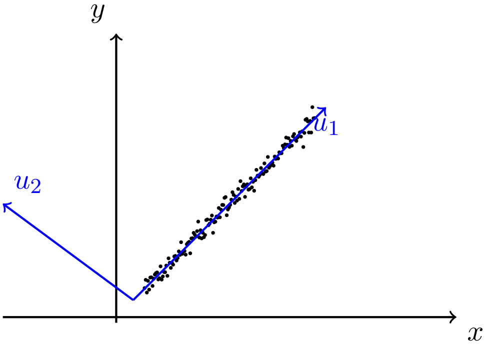
- Consider the following data
- Each point (vector) here is represented using a linear combination of the x and y axes (i.e. using the point's x and y co-ordinates)
- In other words we are using x and y as the basis
- What if we choose a different basis?



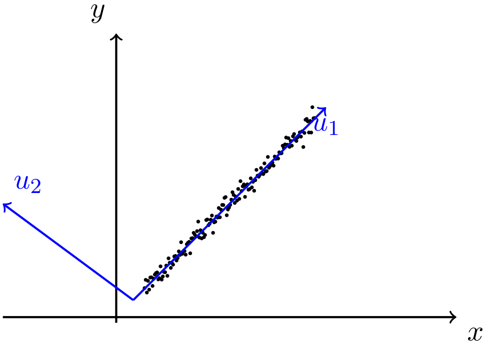
- For example, what if we use u_1 and u_2 as a basis instead of x and y.



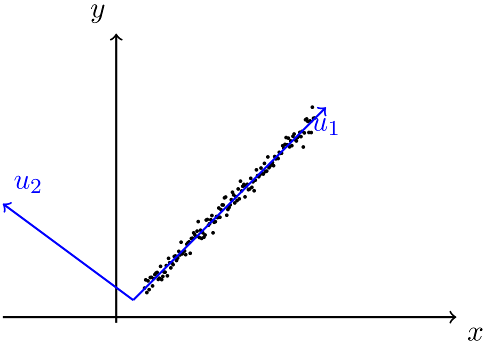
- For example, what if we use u_1 and u_2 as a basis instead of x and y.
- We observe that all the points have a very small component in the direction of u_2 (almost noise)



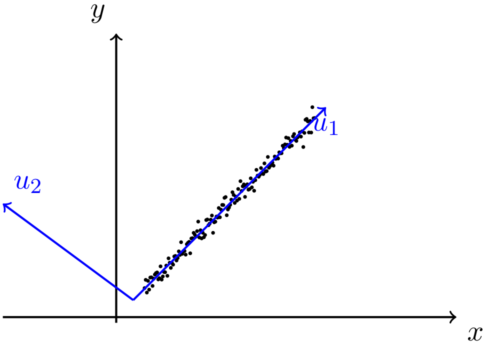
- For example, what if we use u_1 and u_2 as a basis instead of x and y.
- We observe that all the points have a very small component in the direction of u_2 (almost noise)
- It seems that the same data which was originally in $\mathbb{R}^2(x, y)$ can now be represented in $\mathbb{R}^1(u_1)$ by making a smarter choice for the basis



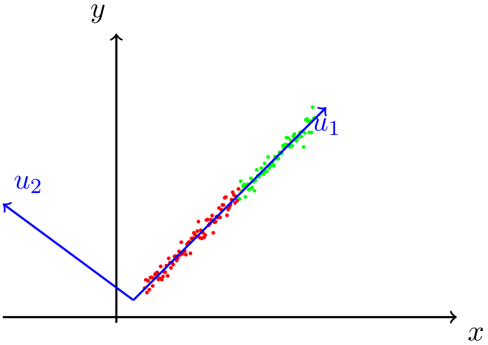
- Let's try stating this more formally



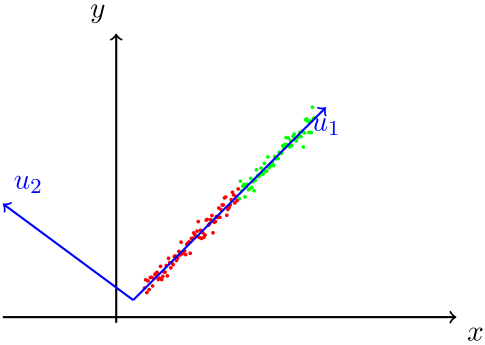
- Let's try stating this more formally
- Why do we not care about u_2?



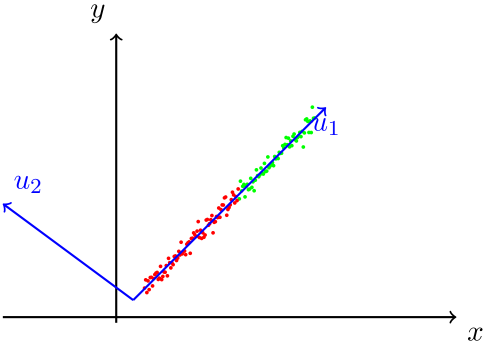
- Let's try stating this more formally
- Why do we not care about u_2?
- Because the variance in the data in this direction is very small (all data points have almost the same value in the u_2 direction)



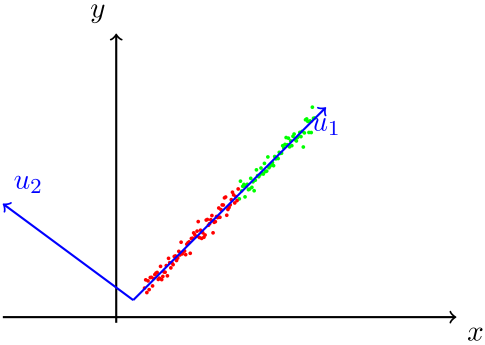
- Let's try stating this more formally
- Why do we not care about u_2?
- Because the variance in the data in this direction is very small (all data points have almost the same value in the u_2 direction)
- If we were to build a classifier on top of this data then u_2 would not contribute to the classifier as the points are not distinguishable along this direction



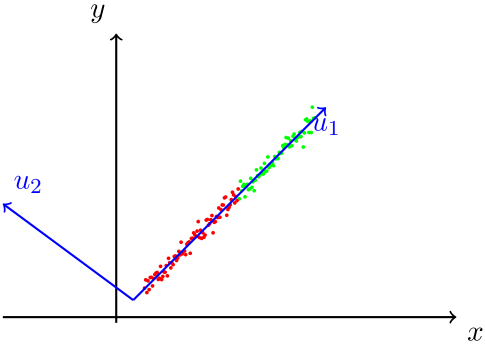
- In general, we are interested in representing the data using fewer dimensions such that



- In general, we are interested in representing the data using fewer dimensions such that the data has high variance along these dimensions



- In general, we are interested in representing the data using fewer dimensions such that the data has high variance along these dimensions
- Is that all?



- In general, we are interested in representing the data using fewer dimensions such that the data has high variance along these dimensions
- Is that all?
- No, there is something else that we desire. Let's see what.

x	y	z
1	1	1
0.5	0	0
0.25	1	1
0.35	1.5	1.5
0.45	1	1
0.57	2	2.1
0.62	1.1	1
0.73	0.75	0.76
0.72	0.86	0.87

- Consider the following data

x	y	z
1	1	1
0.5	0	0
0.25	1	1
0.35	1.5	1.5
0.45	1	1
0.57	2	2.1
0.62	1.1	1
0.73	0.75	0.76
0.72	0.86	0.87

- Consider the following data
- Is z adding any new information beyond what is already contained in y?

x	**y**	**z**
1	1	1
0.5	0	0
0.25	1	1
0.35	1.5	1.5
0.45	1	1
0.57	2	2.1
0.62	1.1	1
0.73	0.75	0.76
0.72	0.86	0.87

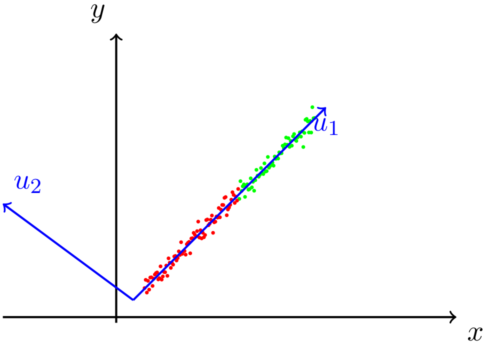
$$\rho_{yz} = \frac{\sum_{i=1}^{n}(y_i - \overline{y})(z_i - \overline{z})}{\sqrt{\sum_{i=1}^{n}(y_i - \overline{y})^2}\sqrt{\sum_{i=1}^{n}(z_i - \overline{z})^2}}$$

- Consider the following data
- Is z adding any new information beyond what is already contained in y?
- The two columns are highly correlated (or they have a high covariance)

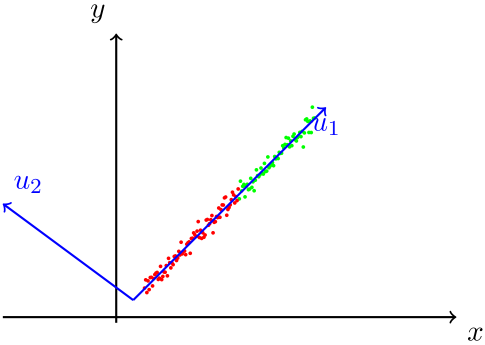
x	**y**	**z**
1	1	1
0.5	0	0
0.25	1	1
0.35	1.5	1.5
0.45	1	1
0.57	2	2.1
0.62	1.1	1
0.73	0.75	0.76
0.72	0.86	0.87

$$\rho_{yz} = \frac{\sum_{i=1}^{n}(y_i - \overline{y})(z_i - \overline{z})}{\sqrt{\sum_{i=1}^{n}(y_i - \overline{y})^2}\sqrt{\sum_{i=1}^{n}(z_i - \overline{z})^2}}$$

- Consider the following data
- Is z adding any new information beyond what is already contained in y?
- The two columns are highly correlated (or they have a high covariance)
- In other words the column z is redundant since it is linearly dependent on y.

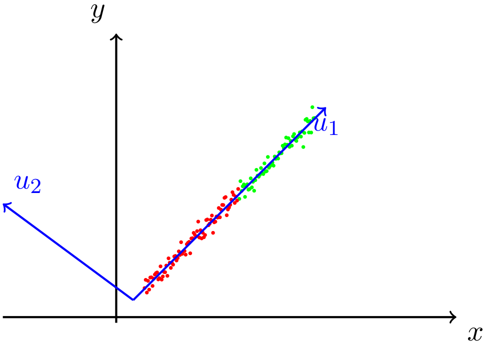


In general, we are interested in representing the data using fewer dimensions such that



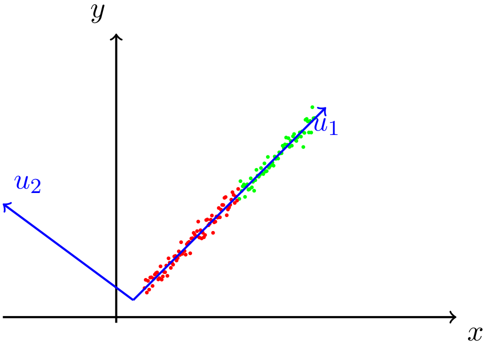
In general, we are interested in representing the data using fewer dimensions such that

- the data has high variance along these dimensions



In general, we are interested in representing the data using fewer dimensions such that

- the data has high variance along these dimensions
- the dimensions are linearly independent (uncorrelated)



In general, we are interested in representing the data using fewer dimensions such that

- the data has high variance along these dimensions
- the dimensions are linearly independent (uncorrelated)
- (even better if they are orthogonal because that is a very convenient basis)

Let $p_1, p_2, \cdots, p_n$ be a set of such n linearly independent orthonormal vectors. Let P be a $n \times n$ matrix such that $p_1, p_2, \cdots, p_n$ are the columns of P.

Let $p_1, p_2, \cdots, p_n$ be a set of such n linearly independent orthonormal vectors. Let P be a $n \times n$ matrix such that $p_1, p_2, \cdots, p_n$ are the columns of P.

Let $x_1, x_2, \cdots, x_m \in \mathbb{R}^n$ be m data points and let X be a matrix such that $x_1, x_2, \cdots, x_m$ are the rows of this matrix. Further let us assume that the data is 0-mean and unit variance.

Let $p_1, p_2, \cdots, p_n$ be a set of such n linearly independent orthonormal vectors. Let P be a $n \times n$ matrix such that $p_1, p_2, \cdots, p_n$ are the columns of P.

Let $x_1, x_2, \cdots, x_m \in \mathbb{R}^n$ be m data points and let X be a matrix such that $x_1, x_2, \cdots, x_m$ are the rows of this matrix. Further let us assume that the data is 0-mean and unit variance.

We want to represent each x_i using this new basis P.

$$x_i = \alpha_{i1}p_1 + \alpha_{i2}p_2 + \alpha_{i3}p_3 + \cdots + \alpha_{in}p_n$$

Let $p_1, p_2, \cdots, p_n$ be a set of such n linearly independent orthonormal vectors. Let P be a $n \times n$ matrix such that $p_1, p_2, \cdots, p_n$ are the columns of P.

Let $x_1, x_2, \cdots, x_m \in \mathbb{R}^n$ be m data points and let X be a matrix such that $x_1, x_2, \cdots, x_m$ are the rows of this matrix. Further let us assume that the data is 0-mean and unit variance.

We want to represent each x_i using this new basis P.

$$x_i = \alpha_{i1}p_1 + \alpha_{i2}p_2 + \alpha_{i3}p_3 + \cdots + \alpha_{in}p_n$$

For an orthonormal basis we know that we can find these $\alpha_i' s$ using

$$\alpha_{ij} = x_i^T p_j = \begin{bmatrix} \leftarrow & x_i & \rightarrow \end{bmatrix}^T \begin{bmatrix} \uparrow \\ p_j \\ \downarrow \end{bmatrix}$$

In general, the transformed data $\hat{x}_i$ is given by

$$\hat{x}_i = \begin{bmatrix} \leftarrow & x_i^T & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & & \uparrow \\ p_1 & \cdots & p_n \\ \downarrow & & \downarrow \end{bmatrix} = x_i^T P$$

In general, the transformed data $\hat{x}_i$ is given by

$$\hat{x}_i = \begin{bmatrix} \leftarrow & x_i^T & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & & \uparrow \\ p_1 & \cdots & p_n \\ \downarrow & & \downarrow \end{bmatrix} = x_i^T P$$

and

$$\hat{X} = XP \qquad (\hat{X} \text{ is the matrix of transformed points})$$

Theorem:

If X is a matrix such that its columns have zero mean and if $\hat{X} = XP$ then the columns of $\hat{X}$ will also have zero mean.

Theorem:

If X is a matrix such that its columns have zero mean and if $\hat{X} = XP$ then the columns of $\hat{X}$ will also have zero mean.

Proof: For any matrix A, $\mathbf{1}^T A$ gives us a row vector with the i^{th} element containing the sum of the i^{th} column of A. (this is easy to see using the row-column picture of matrix multiplication).

Theorem:

If X is a matrix such that its columns have zero mean and if $\hat{X} = XP$ then the columns of $\hat{X}$ will also have zero mean.

Proof: For any matrix A, $\mathbf{1}^T A$ gives us a row vector with the i^{th} element containing the sum of the i^{th} column of A. (this is easy to see using the row-column picture of matrix multiplication).

Consider

$$\mathbf{1}^T \hat{X} = \mathbf{1}^T XP = (\mathbf{1}^T X)P$$

But $\mathbf{1}^T X$ is the row vector containing the sums of the columns of X. Thus $\mathbf{1}^T X = 0$. Therefore, $\mathbf{1}^T \hat{X} = 0$.

Hence the transformed matrix also has columns with sum $= 0$.

Theorem:

If X is a matrix such that its columns have zero mean and if $\hat{X} = XP$ then the columns of $\hat{X}$ will also have zero mean.

Proof: For any matrix A, $\mathbf{1}^T A$ gives us a row vector with the i^{th} element containing the sum of the i^{th} column of A. (this is easy to see using the row-column picture of matrix multiplication).

Consider

$$\mathbf{1}^T \hat{X} = \mathbf{1}^T XP = (\mathbf{1}^T X)P$$

But $\mathbf{1}^T X$ is the row vector containing the sums of the columns of X. Thus $\mathbf{1}^T X = 0$. Therefore, $\mathbf{1}^T \hat{X} = 0$.

Hence the transformed matrix also has columns with sum $= 0$.

Theorem:

$X^T X$ is a symmetric matrix.

Theorem:

If X is a matrix such that its columns have zero mean and if $\hat{X} = XP$ then the columns of $\hat{X}$ will also have zero mean.

Proof: For any matrix A, $\mathbf{1}^T A$ gives us a row vector with the i^{th} element containing the sum of the i^{th} column of A. (this is easy to see using the row-column picture of matrix multiplication).

Consider

$$\mathbf{1}^T \hat{X} = \mathbf{1}^T XP = (\mathbf{1}^T X)P$$

But $\mathbf{1}^T X$ is the row vector containing the sums of the columns of X. Thus $\mathbf{1}^T X = 0$. Therefore, $\mathbf{1}^T \hat{X} = 0$.

Hence the transformed matrix also has columns with sum $= 0$.

Theorem:

$X^T X$ is a symmetric matrix.

Proof: We can write $(X^T X)^T = X^T (X^T)^T = X^T X$

Definition:

If X is a matrix whose columns are zero mean then $\Sigma = \frac{1}{m}X^TX$ is the covariance matrix. In other words each entry Σ_{ij} stores the covariance between columns i and j of X.

Definition:

If X is a matrix whose columns are zero mean then $\Sigma = \frac{1}{m}X^TX$ is the covariance matrix. In other words each entry Σ_{ij} stores the covariance between columns i and j of X.

Explanation: Let C be the covariance matrix of X. Let μ_i, μ_j denote the means of the i^{th} and j^{th} column of X respectively. Then by definition of covariance, we can write :

$$\begin{aligned} C_{ij} &= \frac{1}{m}\sum_{k=1}^{m}(X_{ki}-\mu_i)(X_{kj}-\mu_j) \\ &= \frac{1}{m}\sum_{k=1}^{m}X_{ki}X_{kj} \qquad\qquad (\because \mu_i = \mu_j = 0) \\ &= \frac{1}{m}X_i^TX_j = \frac{1}{m}(X^TX)_{ij} \end{aligned}$$

$$\hat{X} = XP$$

$$\hat{X} = XP$$

- Using the previous theorem & definition, we get $\frac{1}{m}\hat{X}^T\hat{X}$ is the covariance matrix of the transformed data. We can write :

$$\hat{X} = XP$$

- Using the previous theorem & definition, we get $\frac{1}{m}\hat{X}^T\hat{X}$ is the covariance matrix of the transformed data. We can write :

$$\frac{1}{m}\hat{X}^T\hat{X} = \frac{1}{m}(XP)^T XP$$

$$\hat{X} = XP$$

- Using the previous theorem & definition, we get $\frac{1}{m}\hat{X}^T\hat{X}$ is the covariance matrix of the transformed data. We can write :

$$\frac{1}{m}\hat{X}^T\hat{X} = \frac{1}{m}(XP)^T XP = \frac{1}{m}P^T X^T XP = P^T\left(\frac{1}{m}X^T X\right)P$$

$$\hat{X} = XP$$

- Using the previous theorem & definition, we get $\frac{1}{m}\hat{X}^T\hat{X}$ is the covariance matrix of the transformed data. We can write :

$$\frac{1}{m}\hat{X}^T\hat{X} = \frac{1}{m}(XP)^T XP = \frac{1}{m}P^T X^T XP = P^T\left(\frac{1}{m}X^T X\right)P = P^T \Sigma P$$

$$\hat{X} = XP$$

- Using the previous theorem & definition, we get $\frac{1}{m}\hat{X}^T\hat{X}$ is the covariance matrix of the transformed data. We can write :

$$\frac{1}{m}\hat{X}^T\hat{X} = \frac{1}{m}(XP)^T XP = \frac{1}{m}P^T X^T XP = P^T\left(\frac{1}{m}X^T X\right)P = P^T\Sigma P$$

- Each cell i, j of the covariance matrix $\frac{1}{m}\hat{X}^T\hat{X}$ stores the covariance between columns i and j of $\hat{X}$.

$$\hat{X} = XP$$

- Using the previous theorem & definition, we get $\frac{1}{m}\hat{X}^T\hat{X}$ is the covariance matrix of the transformed data. We can write :

$$\frac{1}{m}\hat{X}^T\hat{X} = \frac{1}{m}(XP)^T XP = \frac{1}{m}P^T X^T XP = P^T\left(\frac{1}{m}X^T X\right)P = P^T \Sigma P$$

- Each cell i,j of the covariance matrix $\frac{1}{m}\hat{X}^T\hat{X}$ stores the covariance between columns i and j of $\hat{X}$.
- Ideally we want,

$$\left(\frac{1}{m}\hat{X}^T\hat{X}\right)_{ij} = 0 \qquad i \neq j \text{ (covariance} = 0)$$

$$\left(\frac{1}{m}\hat{X}^T\hat{X}\right)_{ij} \neq 0 \qquad i = j \text{ (variance} \neq 0)$$

$$\hat{X} = XP$$

- Using the previous theorem & definition, we get $\frac{1}{m}\hat{X}^T\hat{X}$ is the covariance matrix of the transformed data. We can write :

$$\frac{1}{m}\hat{X}^T\hat{X} = \frac{1}{m}(XP)^T XP = \frac{1}{m}P^T X^T XP = P^T\left(\frac{1}{m}X^T X\right)P = P^T \Sigma P$$

- Each cell i,j of the covariance matrix $\frac{1}{m}\hat{X}^T\hat{X}$ stores the covariance between columns i and j of $\hat{X}$.
- Ideally we want,

$$\left(\frac{1}{m}\hat{X}^T\hat{X}\right)_{ij} = 0 \qquad i \neq j \text{ (covariance} = 0)$$

$$\left(\frac{1}{m}\hat{X}^T\hat{X}\right)_{ij} \neq 0 \qquad i = j \text{ (variance} \neq 0)$$

In other words, we want

$$\frac{1}{m}\hat{X}^T\hat{X} = P^T\Sigma P = D \qquad [\text{ where D is a diagonal matrix }]$$

- We want,

$$P^T \Sigma P = D$$

- We want,

$$P^T \Sigma P = D$$

- But Σ is a square matrix and P is an orthogonal matrix

- We want,

$$P^T \Sigma P = D$$

- But Σ is a square matrix and P is an orthogonal matrix
- Which orthogonal matrix satisfies the following condition?

- We want,

$$P^T \Sigma P = D$$

- But Σ is a square matrix and P is an orthogonal matrix
- Which orthogonal matrix satisfies the following condition?

$$P^T \Sigma P = D$$

- We want,

$$P^T \Sigma P = D$$

- But Σ is a square matrix and P is an orthogonal matrix
- Which orthogonal matrix satisfies the following condition?

$$P^T \Sigma P = D$$

- In other words, which orthogonal matrix P diagonalizes Σ?

- We want,

$$P^T \Sigma P = D$$

- But Σ is a square matrix and P is an orthogonal matrix
- Which orthogonal matrix satisfies the following condition?

$$P^T \Sigma P = D$$

- In other words, which orthogonal matrix P diagonalizes Σ?
- **Answer:** A matrix P whose columns are the eigen vectors of $\Sigma = X^T X$ [By Eigen Value Decomposition]

- We want,

$$P^T \Sigma P = D$$

- But Σ is a square matrix and P is an orthogonal matrix
- Which orthogonal matrix satisfies the following condition?

$$P^T \Sigma P = D$$

- In other words, which orthogonal matrix P diagonalizes Σ?
- **Answer:** A matrix P whose columns are the eigen vectors of $\Sigma = X^T X$ [By Eigen Value Decomposition]
- Thus, the new basis P used to transform X is the basis consisting of the eigen vectors of $X^T X$

- Why is this a good basis?

- Why is this a good basis?
- Because the eigen vectors of $X^T X$ are linearly independent (**proof : Slide 19 Theorem 1**)

- Why is this a good basis?
- Because the eigen vectors of X^TX are linearly independent (**proof : Slide 19 Theorem 1**)
- And because the eigen vectors of X^TX are orthogonal ($\because X^TX$ is symmetric - **saw proof earlier**)

- Why is this a good basis?
- Because the eigen vectors of X^TX are linearly independent (**proof : Slide 19 Theorem 1**)
- And because the eigen vectors of X^TX are orthogonal ($\because$ X^TX is symmetric - **saw proof earlier**)
- This method is called Principal Component Analysis for transforming the data to a new basis where the dimensions are non-redundant (low covariance) & not noisy (high variance)

- Why is this a good basis?
- Because the eigen vectors of X^TX are linearly independent (**proof : Slide 19 Theorem 1**)
- And because the eigen vectors of X^TX are orthogonal ($\because$ X^TX is symmetric - **saw proof earlier**)
- This method is called Principal Component Analysis for transforming the data to a new basis where the dimensions are non-redundant (low covariance) & not noisy (high variance)
- In practice, we select only the top-k dimensions along which the variance is high (this will become more clear when we look at an alternalte interpretation of PCA)

Module 6.5 : PCA : Interpretation 2

Given n orthogonal linearly independent vectors $P = p_1, p_2, \cdots, p_n$ we can represent x_i exactly as a linear combination of these vectors.

Given n orthogonal linearly independent vectors $P = p_1, p_2, \cdots, p_n$ we can represent x_i exactly as a linear combination of these vectors.

$$x_i = \sum_{j=1}^{n} \alpha_{ij} p_j \quad \text{[we know how to estimate } \alpha'_{ij}s \text{ but we will come back to that later]}$$

Given n orthogonal linearly independent vectors $P = p_1, p_2, \cdots, p_n$ we can represent x_i exactly as a linear combination of these vectors.

$$x_i = \sum_{j=1}^{n} \alpha_{ij} p_j \quad \text{[we know how to estimate } \alpha'_{ij}s \text{ but we will come back to that later]}$$

But we are interested only in the top-k dimensions (we want to get rid of noisy & redundant dimensions)

$$\hat{x}_i = \sum_{j=1}^{k} \alpha_{ik} p_k$$

Given n orthogonal linearly independent vectors $P = p_1, p_2, \cdots, p_n$ we can represent x_i exactly as a linear combination of these vectors.

$$x_i = \sum_{j=1}^{n} \alpha_{ij} p_j \quad \text{[we know how to estimate } \alpha_{ij}'s \text{ but we will come back to that later]}$$

But we are interested only in the top-k dimensions (we want to get rid of noisy & redundant dimensions)

$$\hat{x}_i = \sum_{j=1}^{k} \alpha_{ik} p_k$$

We want to select p_i's such that we minimise the reconstructed error

$$e = \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i)$$

$$e = \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i)$$

$$e = \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} p_j - \sum_{j=1}^{k} \alpha_{ij} p_j \right)^2$$

$$
\begin{aligned}
e &= \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i) \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} p_j - \sum_{j=1}^{k} \alpha_{ij} p_j \right)^2 \\
&= \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^2 = \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^T \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)
\end{aligned}
$$

$$
\begin{aligned}
e &= \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i) \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} p_j - \sum_{j=1}^{k} \alpha_{ij} p_j \right)^2 \\
&= \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^2 = \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^T \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right) \\
&= \sum_{i=1}^{m} \left(\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n \right)^T \left(\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n \right)
\end{aligned}
$$

$$
\begin{aligned}
e &= \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i) \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} p_j - \sum_{j=1}^{k} \alpha_{ij} p_j \right)^2 \\
&= \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^2 = \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^T \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right) \\
&= \sum_{i=1}^{m} (\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n)^T (\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n) \\
&= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij} p_j^T p_j \alpha_{ij} + \sum_{i=1}^{m} \sum_{j=k+1}^{n} \sum_{L=k+1, L \neq k}^{n} \alpha_{ij} p_j^T p_L \alpha_{iL}
\end{aligned}
$$

$$
\begin{aligned}
e &= \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i) \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} p_j - \sum_{j=1}^{k} \alpha_{ij} p_j \right)^2 \\
&= \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^2 = \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^T \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right) \\
&= \sum_{i=1}^{m} (\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n)^T (\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n) \\
&= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij} p_j^T p_j \alpha_{ij} + \sum_{i=1}^{m} \sum_{j=k+1}^{n} \sum_{L=k+1, L\neq k}^{n} \alpha_{ij} p_j^T p_L \alpha_{iL} \\
&= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij}^2 \qquad\qquad (\because p_j^T p_j = 1, p_i^T p_j = 0 \quad \forall i \neq j)
\end{aligned}
$$

$$
\begin{aligned}
e &= \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i) \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} p_j - \sum_{j=1}^{k} \alpha_{ij} p_j \right)^2 \\
&= \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^2 = \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^T \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right) \\
&= \sum_{i=1}^{m} \left(\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n \right)^T \left(\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n \right) \\
&= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij} p_j^T p_j \alpha_{ij} + \sum_{i=1}^{m} \sum_{j=k+1}^{n} \sum_{L=k+1, L \neq k}^{n} \alpha_{ij} p_j^T p_L \alpha_{iL} \\
&= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij}^2 \qquad\qquad (\because p_j^T p_j = 1, p_i^T p_j = 0 \quad \forall i \neq j) \\
&= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \left(x_i^T p_j \right)^2
\end{aligned}
$$

$$= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \left(p_j^T x_i\right)\left(x_i^T p_j\right)$$

$$e = \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} p_j - \sum_{j=1}^{k} \alpha_{ij} p_j \right)^2$$

$$= \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^2 = \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^T \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)$$

$$= \sum_{i=1}^{m} \left(\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n\right)^T \left(\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n\right)$$

$$= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij} p_j^T p_j \alpha_{ij} + \sum_{i=1}^{m} \sum_{j=k+1}^{n} \sum_{L=k+1, L \neq k}^{n} \alpha_{ij} p_j^T p_L \alpha_{iL}$$

$$= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij}^2 \qquad (\because p_j^T p_j = 1, p_i^T p_j = 0 \quad \forall i \neq j)$$

$$= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \left(x_i^T p_j\right)^2$$

$$
\begin{aligned}
e &= \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i) \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} p_j - \sum_{j=1}^{k} \alpha_{ij} p_j \right)^2 \\
&= \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^2 = \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^T \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right) \\
&= \sum_{i=1}^{m} (\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n)^T (\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n) \\
&= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij} p_j^T p_j \alpha_{ij} + \sum_{i=1}^{m} \sum_{j=k+1}^{n} \sum_{L=k+1, L \neq k}^{n} \alpha_{ij} p_j^T p_L \alpha_{iL} \\
&= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij}^2 \qquad (\because p_j^T p_j = 1, p_i^T p_j = 0 \quad \forall i \neq j) \\
&= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \left(x_i^T p_j \right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \left(p_j^T x_i \right) \left(x_i^T p_j \right) \\
&= \sum_{j=k+1}^{n} p_j^T \left(\sum_{i=1}^{m} x_i x_i^T \right) p_j
\end{aligned}
$$

$$e = \sum_{i=1}^{m} (x_i - \hat{x}_i)^T (x_i - \hat{x}_i)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} p_j - \sum_{j=1}^{k} \alpha_{ij} p_j \right)^2$$

$$= \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^2 = \sum_{i=1}^{m} \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)^T \left(\sum_{j=k+1}^{n} \alpha_{ij} p_j \right)$$

$$= \sum_{i=1}^{m} (\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n)^T (\alpha_{i,k+1} p_{k+1} + \alpha_{i,k+2} p_{k+2} + \ldots + \alpha_{i,n} p_n)$$

$$= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij} p_j^T p_j \alpha_{ij} + \sum_{i=1}^{m} \sum_{j=k+1}^{n} \sum_{L=k+1, L \neq k}^{n} \alpha_{ij} p_j^T p_L \alpha_{iL}$$

$$= \sum_{i=1}^{m} \sum_{j=k+1}^{n} \alpha_{ij}^2 \qquad (\because p_j^T p_j = 1, p_i^T p_j = 0 \quad \forall i \neq j)$$

$$= \sum_{i=1}^{m} \sum_{j=k+1}^{n} (x_i^T p_j)^2$$

$$= \sum_{i=1}^{m} \sum_{j=k+1}^{n} (p_j^T x_i)(x_i^T p_j)$$

$$= \sum_{j=k+1}^{n} p_j^T \left(\sum_{i=1}^{m} x_i x_i^T \right) p_j$$

$$= \sum_{j=k+1}^{n} p_j^T m C p_j \qquad \left[\because \frac{1}{m} \sum_{i=1}^{m} x_i x_i^T = \frac{X^T X}{m} = C \right]$$

We want to minimize e

$$\min_{p_{k+1}, p_{k+2}, \cdots, p_n} \sum_{j=k+1}^{n} p_j^T m C p_j \qquad s.t. \quad p_j^T p_j = 1 \quad \forall j = k+1, k+2, \cdots, n$$

We want to minimize e

$$\min_{p_{k+1},p_{k+2},\cdots,p_n} \sum_{j=k+1}^{n} p_j^T mCp_j \qquad s.t. \quad p_j^T p_j = 1 \quad \forall j = k+1, k+2, \cdots, n$$

The solution to the above problem is given by the eigen vectors corresponding to the smallest eigen values of C (**Proof : refer Slide 26**).

We want to minimize e

$$\min_{p_{k+1}, p_{k+2}, \cdots, p_n} \sum_{j=k+1}^{n} p_j^T mC p_j \qquad s.t. \quad p_j^T p_j = 1 \quad \forall j = k+1, k+2, \cdots, n$$

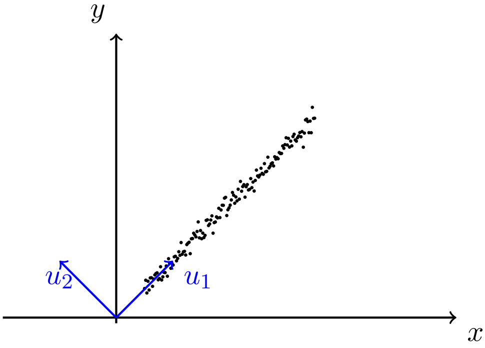
The solution to the above problem is given by the eigen vectors corresponding to the smallest eigen values of C (**Proof : refer Slide 26**).

Thus we select $P = p_1, p_2, \cdots, p_n$ as eigen vectors of C and retain only top-k eigen vectors to express the data [or discard the eigen vectors $k+1, \cdots, n$]

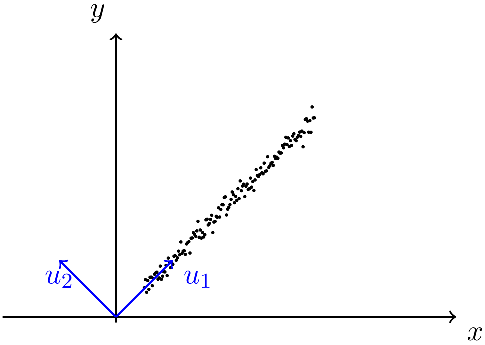
Key Idea

Minimize the error in reconstructing x_i after projecting the data on to a new basis.

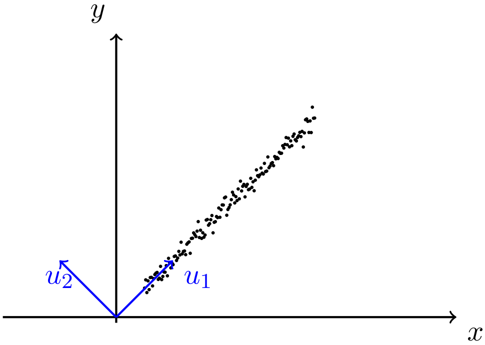
Let's look at the ***'Reconstruction Error'*** *in the context of our toy example*



- $u_1 = [1,1]$ and $u_2 = [-1,1]$ are the new basis vectors

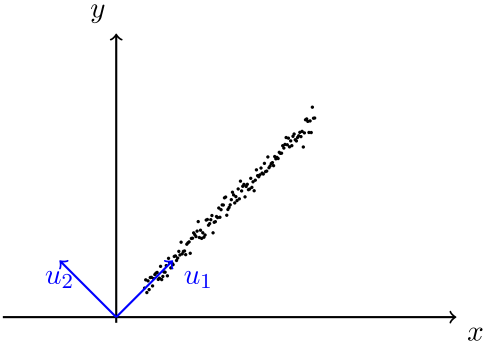


- $u_1 = [1, 1]$ and $u_2 = [-1, 1]$ are the new basis vectors
- Let us convert them to unit vectors $u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ & $u_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$



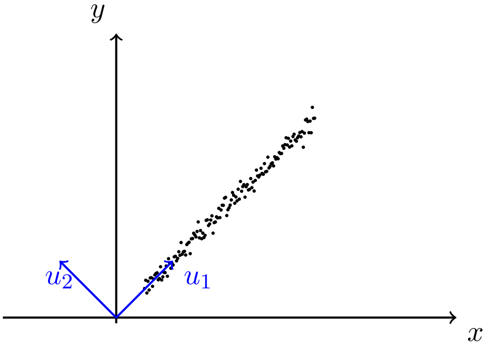
- $u_1 = [1,1]$ and $u_2 = [-1,1]$ are the new basis vectors
- Let us convert them to unit vectors $u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ & $u_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

- Consider the point $x = [3.3, 3]$ in the original data



- $u_1 = [1, 1]$ and $u_2 = [-1, 1]$ are the new basis vectors
- Let us convert them to unit vectors $u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ & $u_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

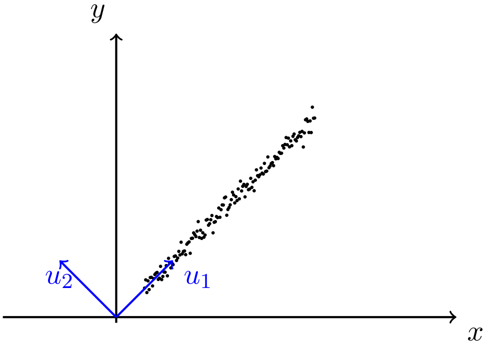
- Consider the point $x = [3.3, 3]$ in the original data
- $\alpha_1 = x^T u_1 = 6.3/\sqrt{2}$
  $\alpha_2 = x^T u_2 = -0.3/\sqrt{2}$



- $u_1 = [1, 1]$ and $u_2 = [-1, 1]$ are the new basis vectors
- Let us convert them to unit vectors $u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ & $u_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

- Consider the point $x = [3.3, 3]$ in the original data
- $\alpha_1 = x^T u_1 = 6.3/\sqrt{2}$
  $\alpha_2 = x^T u_2 = -0.3/\sqrt{2}$
- the perfect reconstruction of x is given by (using $n = 2$ dimensions)

$$x = \alpha_1 u_1 + \alpha_2 u_2 = \begin{bmatrix} 3.3 & 3 \end{bmatrix}$$



- $u_1 = [1, 1]$ and $u_2 = [-1, 1]$ are the new basis vectors
- Let us convert them to unit vectors $u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ & $u_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

- Consider the point $x = [3.3, 3]$ in the original data
- $\alpha_1 = x^T u_1 = 6.3/\sqrt{2}$
  $\alpha_2 = x^T u_2 = -0.3/\sqrt{2}$
- the perfect reconstruction of x is given by (using $n = 2$ dimensions)

$$x = \alpha_1 u_1 + \alpha_2 u_2 = \begin{bmatrix} 3.3 & 3 \end{bmatrix}$$

- But we are going to reconstruct it using fewer (only $k = 1 < n$ dimensions, ignoring the low variance u_2 dimension)

$$\hat{x} = \alpha_1 u_1 = \begin{bmatrix} 3.15 & 3.15 \end{bmatrix}$$

(reconstruction with minimum error)

Recap

- The eigen vectors of a matrix with distinct eigenvalues are linearly independent

Recap

- The eigen vectors of a matrix with distinct eigenvalues are linearly independent
- The eigen vectors of a square symmetric matrix are orthogonal

Recap

- The eigen vectors of a matrix with distinct eigenvalues are linearly independent
- The eigen vectors of a square symmetric matrix are orthogonal
- PCA exploits this fact by representing the data using a new basis comprising only the top-k eigen vectors

Recap

- The eigen vectors of a matrix with distinct eigenvalues are linearly independent
- The eigen vectors of a square symmetric matrix are orthogonal
- PCA exploits this fact by representing the data using a new basis comprising only the top-k eigen vectors
- The $n - k$ dimensions which contribute very little to the reconstruction error are discarded

Recap

- The eigen vectors of a matrix with distinct eigenvalues are linearly independent
- The eigen vectors of a square symmetric matrix are orthogonal
- PCA exploits this fact by representing the data using a new basis comprising only the top-k eigen vectors
- The $n - k$ dimensions which contribute very little to the reconstruction error are discarded
- **These are also the directions along which the variance is minimum**

Module 6.6 : PCA : Interpretation 3

- We started off with the following wishlist

- We started off with the following wishlist
- We are interested in representing the data using fewer dimensions such that

- We started off with the following wishlist
- We are interested in representing the data using fewer dimensions such that
  - the dimensions have low covariance

- We started off with the following wishlist
- We are interested in representing the data using fewer dimensions such that
  - the dimensions have low covariance
  - the dimensions have high variance

- We started off with the following wishlist
- We are interested in representing the data using fewer dimensions such that
  - the dimensions have low covariance
  - the dimensions have high variance
- So far we have paid a lot of attention to the covariance

- We started off with the following wishlist
- We are interested in representing the data using fewer dimensions such that
  - the dimensions have low covariance
  - the dimensions have high variance
- So far we have paid a lot of attention to the covariance
- It has indeed played a central role in all our analysis

- We started off with the following wishlist
- We are interested in representing the data using fewer dimensions such that
  - the dimensions have low covariance
  - the dimensions have high variance
- So far we have paid a lot of attention to the covariance
- It has indeed played a central role in all our analysis
- But what about variance? Have we achieved our stated goal of high variance along dimensions?

- We started off with the following wishlist
- We are interested in representing the data using fewer dimensions such that
  - the dimensions have low covariance
  - the dimensions have high variance
- So far we have paid a lot of attention to the covariance
- It has indeed played a central role in all our analysis
- But what about variance? Have we achieved our stated goal of high variance along dimensions?
- To answer this question we will see yet another interpretation of PCA

The i^{th} dimension of the transformed data $\hat{X}$ is given by

$$\hat{X}_i = Xp_i$$

The i^{th} dimension of the transformed data $\hat{X}$ is given by

$$\hat{X}_i = Xp_i$$

The variance along this dimension is given by

The i^{th} dimension of the transformed data $\hat{X}$ is given by

$$\hat{X}_i = Xp_i$$

The variance along this dimension is given by

$$\frac{\hat{X}_i^T \hat{X}_i}{m} = \frac{1}{m} p_i^T \underbrace{X^T X p_i}$$

The i^{th} dimension of the transformed data $\hat{X}$ is given by

$$\hat{X}_i = X p_i$$

The variance along this dimension is given by

$$\begin{aligned}
\frac{\hat{X}_i^T \hat{X}_i}{m} &= \frac{1}{m} p_i^T \underbrace{X^T X p_i} \\
&= \frac{1}{m} p_i^T \lambda_i p_i \qquad\qquad [\because p_i \text{ is the eigen vector of } X^T X]
\end{aligned}$$

The i^{th} dimension of the transformed data $\hat{X}$ is given by

$$\hat{X}_i = Xp_i$$

The variance along this dimension is given by

$$\begin{aligned}
\frac{\hat{X}_i^T \hat{X}_i}{m} &= \frac{1}{m} p_i^T \underbrace{X^T X p_i} \\
&= \frac{1}{m} p_i^T \lambda_i p_i \qquad\qquad [\because p_i \text{ is the eigen vector of } X^T X] \\
&= \frac{1}{m} \lambda_i \underbrace{p_i^T p_i}_{=1}
\end{aligned}$$

The i^{th} dimension of the transformed data $\hat{X}$ is given by

$$\hat{X}_i = X p_i$$

The variance along this dimension is given by

$$\begin{aligned}
\frac{\hat{X}_i^T \hat{X}_i}{m} &= \frac{1}{m} p_i^T \underbrace{X^T X p_i} \\
&= \frac{1}{m} p_i^T \lambda_i p_i \qquad [\because p_i \text{ is the eigen vector of } X^T X] \\
&= \frac{1}{m} \lambda_i \underbrace{p_i^T p_i}_{=1} \\
&= \frac{\lambda_i}{m}
\end{aligned}$$

The i^{th} dimension of the transformed data $\hat{X}$ is given by

$$\hat{X}_i = X p_i$$

The variance along this dimension is given by

$$\begin{aligned}
\frac{\hat{X}_i^T \hat{X}_i}{m} &= \frac{1}{m} p_i^T \underbrace{X^T X p_i} \\
&= \frac{1}{m} p_i^T \lambda_i p_i \qquad\qquad [\because p_i \text{ is the eigen vector of } X^T X] \\
&= \frac{1}{m} \lambda_i \underbrace{p_i^T p_i}_{=1} \\
&= \frac{\lambda_i}{m}
\end{aligned}$$

- Thus the variance along the i^{th} dimension (i^{th} eigen vector of $X^T X$) is given by the corresponding (scaled) eigen value.

The i^{th} dimension of the transformed data $\hat{X}$ is given by

$$\hat{X}_i = Xp_i$$

The variance along this dimension is given by

$$\begin{aligned}
\frac{\hat{X}_i^T \hat{X}_i}{m} &= \frac{1}{m} p_i^T \underbrace{X^T X p_i} \\
&= \frac{1}{m} p_i^T \lambda_i p_i \qquad [\because p_i \text{ is the eigen vector of } X^T X] \\
&= \frac{1}{m} \lambda_i \underbrace{p_i^T p_i}_{=1} \\
&= \frac{\lambda_i}{m}
\end{aligned}$$

- Thus the variance along the i^{th} dimension (i^{th} eigen vector of $X^T X$) is given by the corresponding (scaled) eigen value.
- Hence, we did the right thing by discarding the dimensions (eigenvectors) corresponding to lower eigen values!

A Quick Summary

We have seen 3 different interpretations of PCA

A Quick Summary

We have seen 3 different interpretations of PCA

- It ensures that the covariance between the new dimensions is minimized

A Quick Summary

We have seen 3 different interpretations of PCA

- It ensures that the covariance between the new dimensions is minimized
- It picks up dimensions such that the data exhibits a high variance across these dimensions

A Quick Summary

We have seen 3 different interpretations of PCA

- It ensures that the covariance between the new dimensions is minimized
- It picks up dimensions such that the data exhibits a high variance across these dimensions
- It ensures that the data can be represented using less number of dimensions

Module 6.7 : PCA : Practical Example

- Consider we are given a large number of images of human faces (say, m images)

- Consider we are given a large number of images of human faces (say, m images)
- Each image is 100×100 [10K dimensions]

- Consider we are given a large number of images of human faces (say, m images)
- Each image is 100×100 [10K dimensions]
- We would like to represent and store the images using much fewer dimensions (around 50-200)

- Consider we are given a large number of images of human faces (say, m images)
- Each image is 100×100 [10K dimensions]
- We would like to represent and store the images using much fewer dimensions (around 50-200)
- We construct a matrix $X \in \mathbb{R}^{m \times 10K}$

- Consider we are given a large number of images of human faces (say, m images)
- Each image is 100×100 [10K dimensions]
- We would like to represent and store the images using much fewer dimensions (around 50-200)
- We construct a matrix $X \in \mathbb{R}^{m \times 10K}$
- Each row of the matrix corresponds to 1 image

- Consider we are given a large number of images of human faces (say, m images)
- Each image is 100×100 [10K dimensions]
- We would like to represent and store the images using much fewer dimensions (around 50-200)
- We construct a matrix $X \in \mathbb{R}^{m \times 10K}$
- Each row of the matrix corresponds to 1 image
- Each image is represented using 10K dimensions

- $X \in \mathbb{R}^{m \times 10K}$ (as explained on the previous slide)

- $X \in \mathbb{R}^{m \times 10K}$ (as explained on the previous slide)
- We retain the top 100 dimensions corresponding to the top 100 eigen vectors of $X^T X$

- $X \in \mathbb{R}^{m \times 10K}$ (as explained on the previous slide)
- We retain the top 100 dimensions corresponding to the top 100 eigen vectors of $X^T X$
- Note that $X^T X$ is a $n \times n$ matrix so its eigen vectors will be n dimensional ($n = 10K$ in this case)

- $X \in \mathbb{R}^{m \times 10K}$ (as explained on the previous slide)
- We retain the top 100 dimensions corresponding to the top 100 eigen vectors of X^TX
- Note that X^TX is a $n \times n$ matrix so its eigen vectors will be n dimensional ($n = 10K$ in this case)
- We can convert each eigen vector into a 100×100 matrix and treat it as an image

- $X \in \mathbb{R}^{m \times 10K}$ (as explained on the previous slide)
- We retain the top 100 dimensions corresponding to the top 100 eigen vectors of $X^T X$
- Note that $X^T X$ is a $n \times n$ matrix so its eigen vectors will be n dimensional ($n = 10K$ in this case)
- We can convert each eigen vector into a 100×100 matrix and treat it as an image
- Let's see what we get

- $X \in \mathbb{R}^{m \times 10K}$ (as explained on the previous slide)
- We retain the top 100 dimensions corresponding to the top 100 eigen vectors of X^TX
- Note that X^TX is a $n \times n$ matrix so its eigen vectors will be n dimensional ($n = 10K$ in this case)
- We can convert each eigen vector into a 100×100 matrix and treat it as an image
- Let's see what we get
- What we have plotted here are the first 16 eigen vectors of X^TX (basically, treating each 10K dimensional eigen vector as a 100×100 dimensional image)

- These images are called eigenfaces and form a basis for representing any face in our database

- These images are called eigenfaces and form a basis for representing any face in our database
- In other words, we can now represent a given image (face) as a linear combination of these eigen faces

- These images are called eigenfaces and form a basis for representing any face in our database
- In other words, we can now represent a given image (face) as a linear combination of these eigen faces

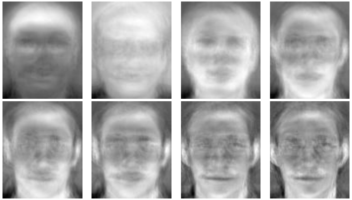
$$\sum_{i=1}^{1} \alpha_{1i} p_i$$

- These images are called eigenfaces and form a basis for representing any face in our database
- In other words, we can now represent a given image (face) as a linear combination of these eigen faces

$$\sum_{i=1}^{2} \alpha_{1i} p_i$$

- These images are called eigenfaces and form a basis for representing any face in our database
- In other words, we can now represent a given image (face) as a linear combination of these eigen faces

$$\sum_{i=1}^{4} \alpha_{1i} p_i$$



- These images are called eigenfaces and form a basis for representing any face in our database
- In other words, we can now represent a given image (face) as a linear combination of these eigen faces

$$\sum_{i=1}^{8} \alpha_{1i} p_i$$

- These images are called eigenfaces and form a basis for representing any face in our database
- In other words, we can now represent a given image (face) as a linear combination of these eigen faces

$$\sum_{i=1}^{12} \alpha_{1i} p_i$$

$$\sum_{i=1}^{16} \alpha_{1i} p_i$$

- These images are called eigenfaces and form a basis for representing any face in our database
- In other words, we can now represent a given image (face) as a linear combination of these eigen faces

$$\sum_{i=1}^{16} \alpha_{1i} p_i$$

- These images are called eigenfaces and form a basis for representing any face in our database
- In other words, we can now represent a given image (face) as a linear combination of these eigen faces
- In practice, we just need to store $p_1, p_2, \cdots, p_k$ (one time storage)

$$\sum_{i=1}^{16} \alpha_{1i} p_i$$

- These images are called eigenfaces and form a basis for representing any face in our database
- In other words, we can now represent a given image (face) as a linear combination of these eigen faces
- In practice, we just need to store $p_1, p_2, \cdots, p_k$ (one time storage)
- Then for each image i we just need to store the scalar values $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ik}$

$$\sum_{i=1}^{16} \alpha_{1i} p_i$$

- These images are called eigenfaces and form a basis for representing any face in our database
- In other words, we can now represent a given image (face) as a linear combination of these eigen faces
- In practice, we just need to store $p_1, p_2, \cdots, p_k$ (one time storage)
- Then for each image i we just need to store the scalar values $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ik}$
- This significantly reduces the storage cost without much loss in image quality

Module 6.8 : Singular Value Decomposition

Let us get some more perspective on eigen vectors before moving ahead

- Let $v_1, v_2, \cdots, v_n$ be the eigen vectors of A and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be corresponding eigen values

$$Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2, \cdots, Av_n = \lambda_n v_n$$

- Let $v_1, v_2, \cdots, v_n$ be the eigen vectors of A and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be corresponding eigen values

$$Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2, \cdots, Av_n = \lambda_n v_n$$

- If a vector x in $\mathbb{R}^n$ is represented using $v_1, v_2, \cdots, v_n$ as basis then

$$x = \sum_{i=1}^{n} \alpha_i v_i$$

- Let $v_1, v_2, \cdots, v_n$ be the eigen vectors of A and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be corresponding eigen values

$$Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2, \cdots, Av_n = \lambda_n v_n$$

- If a vector x in $\mathbb{R}^n$ is represented using $v_1, v_2, \cdots, v_n$ as basis then

$$x = \sum_{i=1}^{n} \alpha_i v_i$$

$$\text{Now, } Ax = \sum_{i=1}^{n} \alpha_i A v_i = \sum_{i=1}^{n} \alpha_i \lambda_i v_i$$

- Let $v_1, v_2, \cdots, v_n$ be the eigen vectors of A and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be corresponding eigen values

$$Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2, \cdots, Av_n = \lambda_n v_n$$

- If a vector x in $\mathbb{R}^n$ is represented using $v_1, v_2, \cdots, v_n$ as basis then

$$x = \sum_{i=1}^{n} \alpha_i v_i$$

$$\text{Now, } Ax = \sum_{i=1}^{n} \alpha_i A v_i = \sum_{i=1}^{n} \alpha_i \lambda_i v_i$$

- The matrix multiplication reduces to a scalar multiplication if the eigen vectors of A are used as a basis.

- So far all the discussion was centered around square matrices ($A \in \mathbb{R}^{n \times n}$)

- So far all the discussion was centered around square matrices ($A \in \mathbb{R}^{n \times n}$)
- What about rectangular matrices $A \in \mathbb{R}^{m \times n}$? Can they have eigen vectors?

- So far all the discussion was centered around square matrices ($A \in \mathbb{R}^{n \times n}$)
- What about rectangular matrices $A \in \mathbb{R}^{m \times n}$? Can they have eigen vectors?
- Is it possible to have $A_{m \times n} x_{n \times 1} = x_{n \times 1}$?

- So far all the discussion was centered around square matrices ($A \in \mathbb{R}^{n\times n}$)
- What about rectangular matrices $A \in \mathbb{R}^{m\times n}$? Can they have eigen vectors?
- Is it possible to have $A_{m\times n}x_{n\times 1} = x_{n\times 1}$? Not possible !

- So far all the discussion was centered around square matrices ($A \in \mathbb{R}^{n \times n}$)
- What about rectangular matrices $A \in \mathbb{R}^{m \times n}$? Can they have eigen vectors?
- Is it possible to have $A_{m \times n} x_{n \times 1} = x_{n \times 1}$? Not possible !
- The result of $A_{m \times n} x_{n \times 1}$ is a vector belonging to $\mathbb{R}^m$ (whereas $x \in \mathbb{R}^n$)

- So far all the discussion was centered around square matrices ($A \in \mathbb{R}^{n \times n}$)
- What about rectangular matrices $A \in \mathbb{R}^{m \times n}$? Can they have eigen vectors?
- Is it possible to have $A_{m \times n} x_{n \times 1} = x_{n \times 1}$? Not possible !
- The result of $A_{m \times n} x_{n \times 1}$ is a vector belonging to $\mathbb{R}^m$ (whereas $x \in \mathbb{R}^n$)
- So do we miss out on the advantage that a basis of eigen vectors provides for square matrices (i.e. converting matrix multiplications into scalar multiplications)?

- So far all the discussion was centered around square matrices ($A \in \mathbb{R}^{n \times n}$)
- What about rectangular matrices $A \in \mathbb{R}^{m \times n}$? Can they have eigen vectors?
- Is it possible to have $A_{m \times n} x_{n \times 1} = x_{n \times 1}$? Not possible !
- The result of $A_{m \times n} x_{n \times 1}$ is a vector belonging to $\mathbb{R}^m$ (whereas $x \in \mathbb{R}^n$)
- So do we miss out on the advantage that a basis of eigen vectors provides for square matrices (i.e. converting matrix multiplications into scalar multiplications)?
- We will see the answer to this question over the next few slides

- Note that matrix $A_{m \times n}$ provides a transformation $\mathbb{R}^n \rightarrow \mathbb{R}^m$

- Note that matrix $A_{m \times n}$ provides a transformation $\mathbb{R}^n \to \mathbb{R}^m$
- What if we could have pairs of vectors $(v_1, u_1), (v_2, u_2), \cdots, (v_k, u_k)$ such that $v_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $Av_i = \sigma_i u_i$

- Note that matrix $A_{m \times n}$ provides a transformation $\mathbb{R}^n \rightarrow \mathbb{R}^m$
- What if we could have pairs of vectors $(v_1, u_1), (v_2, u_2), \cdots, (v_k, u_k)$ such that $v_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $Av_i = \sigma_i u_i$
- Further let's assume that $v_1, \cdots, v_k, \cdots, v_n$ are orthogonal & thus form a basis V in $\mathbb{R}^n$

- Note that matrix $A_{m \times n}$ provides a transformation $\mathbb{R}^n \to \mathbb{R}^m$
- What if we could have pairs of vectors $(v_1, u_1), (v_2, u_2), \cdots, (v_k, u_k)$ such that $v_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $Av_i = \sigma_i u_i$
- Further let's assume that $v_1, \cdots, v_k, \cdots, v_n$ are orthogonal & thus form a basis V in $\mathbb{R}^n$
- Similarly let's assume that $u_1, \cdots, u_k, \cdots, u_m$ are orthogonal & thus form a basis U in $\mathbb{R}^m$

- Note that matrix $A_{m\times n}$ provides a transformation $\mathbb{R}^n \rightarrow \mathbb{R}^m$
- What if we could have pairs of vectors $(v_1, u_1), (v_2, u_2), \cdots, (v_k, u_k)$ such that $v_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $Av_i = \sigma_i u_i$
- Further let's assume that $v_1, \cdots, v_k, \cdots, v_n$ are orthogonal & thus form a basis V in $\mathbb{R}^n$
- Similarly let's assume that $u_1, \cdots, u_k, \cdots, u_m$ are orthogonal & thus form a basis U in $\mathbb{R}^m$
- Now what if every vector $x \in \mathbb{R}^n$ is represented using the basis V

- Note that matrix $A_{m\times n}$ provides a transformation $\mathbb{R}^n \to \mathbb{R}^m$
- What if we could have pairs of vectors $(v_1, u_1), (v_2, u_2), \cdots, (v_k, u_k)$ such that $v_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $Av_i = \sigma_i u_i$
- Further let's assume that $v_1, \cdots, v_k, \cdots, v_n$ are orthogonal & thus form a basis V in $\mathbb{R}^n$
- Similarly let's assume that $u_1, \cdots, u_k, \cdots, u_m$ are orthogonal & thus form a basis U in $\mathbb{R}^m$
- Now what if every vector $x \in \mathbb{R}^n$ is represented using the basis V

$$x = \sum_{i=1}^{k} \alpha_i v_i$$

[note we are using k instead of n ; will clarify this in a minute]

- Note that matrix $A_{m \times n}$ provides a transformation $\mathbb{R}^n \to \mathbb{R}^m$
- What if we could have pairs of vectors $(v_1, u_1), (v_2, u_2), \cdots, (v_k, u_k)$ such that $v_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $Av_i = \sigma_i u_i$
- Further let's assume that $v_1, \cdots, v_k, \cdots, v_n$ are orthogonal & thus form a basis V in $\mathbb{R}^n$
- Similarly let's assume that $u_1, \cdots, u_k, \cdots, u_m$ are orthogonal & thus form a basis U in $\mathbb{R}^m$
- Now what if every vector $x \in \mathbb{R}^n$ is represented using the basis V

$$x = \sum_{i=1}^{k} \alpha_i v_i$$

[note we are using k instead of n ; will clarify this in a minute]

$$Ax = \sum_{i=1}^{k} \alpha_i A v_i$$

- Note that matrix $A_{m \times n}$ provides a transformation $\mathbb{R}^n \to \mathbb{R}^m$
- What if we could have pairs of vectors $(v_1, u_1), (v_2, u_2), \cdots, (v_k, u_k)$ such that $v_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $Av_i = \sigma_i u_i$
- Further let's assume that $v_1, \cdots, v_k, \cdots, v_n$ are orthogonal & thus form a basis V in $\mathbb{R}^n$
- Similarly let's assume that $u_1, \cdots, u_k, \cdots, u_m$ are orthogonal & thus form a basis U in $\mathbb{R}^m$
- Now what if every vector $x \in \mathbb{R}^n$ is represented using the basis V

$$x = \sum_{i=1}^{k} \alpha_i v_i$$

[note we are using k instead of n ; will clarify this in a minute]

$$\begin{aligned} Ax &= \sum_{i=1}^{k} \alpha_i A v_i \\ &= \sum_{i=1}^{k} \alpha_i \sigma_i u_i \end{aligned}$$

- Note that matrix $A_{m \times n}$ provides a transformation $\mathbb{R}^n \to \mathbb{R}^m$
- What if we could have pairs of vectors $(v_1, u_1), (v_2, u_2), \cdots, (v_k, u_k)$ such that $v_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $Av_i = \sigma_i u_i$
- Further let's assume that $v_1, \cdots, v_k, \cdots, v_n$ are orthogonal & thus form a basis V in $\mathbb{R}^n$
- Similarly let's assume that $u_1, \cdots, u_k, \cdots, u_m$ are orthogonal & thus form a basis U in $\mathbb{R}^m$
- Now what if every vector $x \in \mathbb{R}^n$ is represented using the basis V

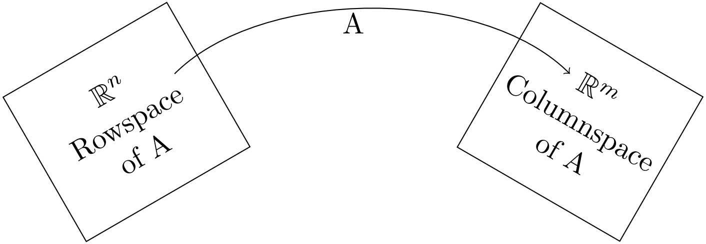
$$x = \sum_{i=1}^{k} \alpha_i v_i \qquad \text{[note we are using } k \text{ instead of } n \text{ ; will clarify this in a minute]}$$

$$Ax = \sum_{i=1}^{k} \alpha_i A v_i$$

$$= \sum_{i=1}^{k} \alpha_i \sigma_i u_i$$

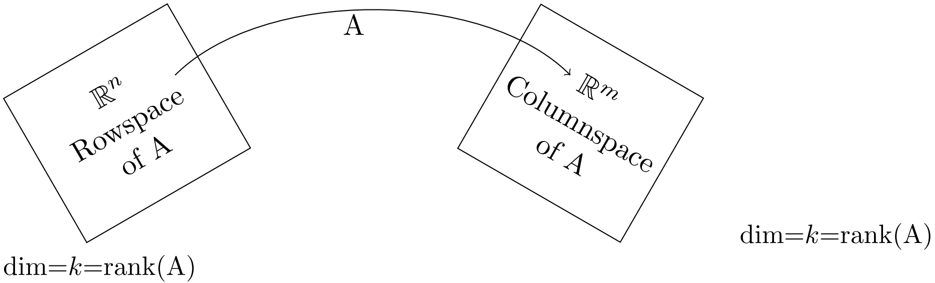
- Once again the matrix multiplication reduces to a scalar multiplication

Let's look at a geometric interpretation of this

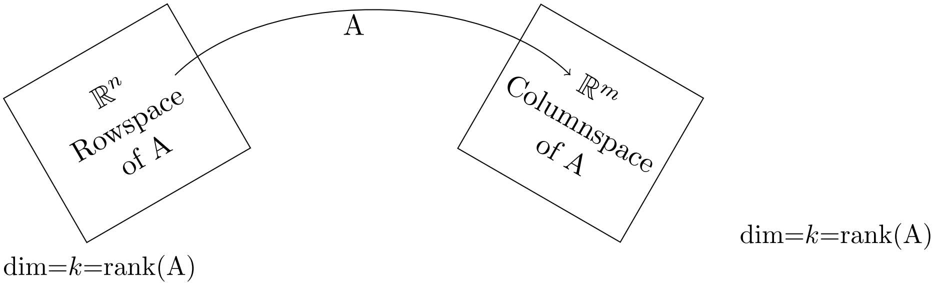


dim=k=rank(A)

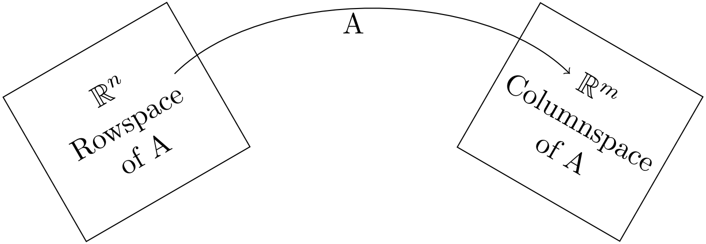
dim=k=rank(A)



- $\mathbb{R}^n$ - Space of all vectors which can multiply with A to give Ax [this is the space of inputs of the function]



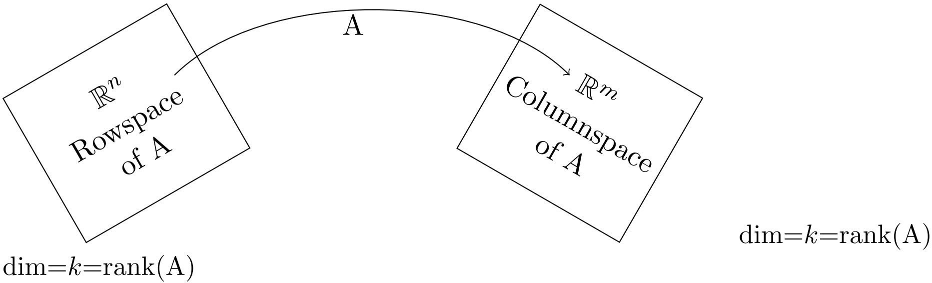
- $\mathbb{R}^n$ - Space of all vectors which can multiply with A to give Ax [this is the space of inputs of the function]
- $\mathbb{R}^m$ - Space of all vectors which are outputs of the function Ax



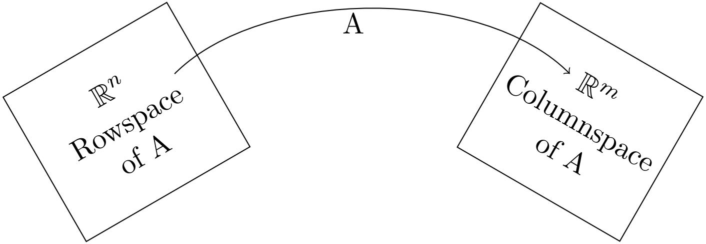
dim=k=rank(A)

dim=k=rank(A)

- $\mathbb{R}^n$ - Space of all vectors which can multiply with A to give Ax [this is the space of inputs of the function]
- $\mathbb{R}^m$ - Space of all vectors which are outputs of the function Ax
- We are interested in finding a basis U, V such that



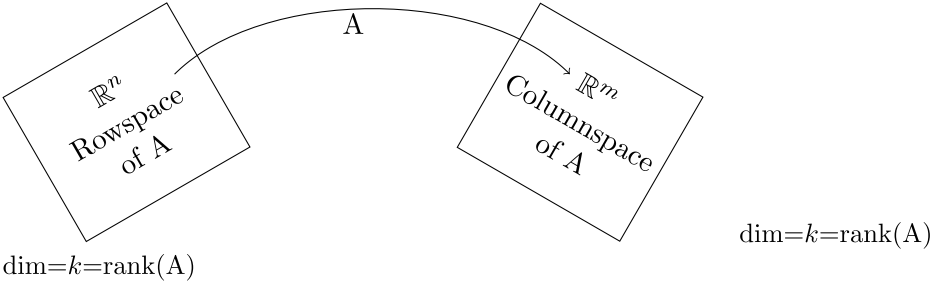
- $\mathbb{R}^n$ - Space of all vectors which can multiply with A to give Ax [this is the space of inputs of the function]
- $\mathbb{R}^m$ - Space of all vectors which are outputs of the function Ax
- We are interested in finding a basis U, V such that
  - V - basis for inputs



dim=k=rank(A)

dim=k=rank(A)

- $\mathbb{R}^n$ - Space of all vectors which can multiply with A to give Ax [this is the space of inputs of the function]
- $\mathbb{R}^m$ - Space of all vectors which are outputs of the function Ax
- We are interested in finding a basis U, V such that
    - V - basis for inputs
    - U - basis for outputs



- $\mathbb{R}^n$ - Space of all vectors which can multiply with A to give Ax [this is the space of inputs of the function]
- $\mathbb{R}^m$ - Space of all vectors which are outputs of the function Ax
- We are interested in finding a basis U, V such that
  - V - basis for inputs
  - U - basis for outputs
- such that if the inputs and outputs are represented using this basis then the operation Ax reduces to a scalar operation

- What do we mean by saying that dimension of rowspace is k? If $x \in \mathbb{R}^n$ then why is the dimension not n.

- What do we mean by saying that dimension of rowspace is k? If $x \in \mathbb{R}^n$ then why is the dimension not n.
- It means that of all the possible vectors in $\mathbb{R}^n$ only a subspace of vectors lying in $\mathbb{R}^k$ can act as inputs to Ax and produce a non-zero output. The remaining vectors in $\mathbb{R}^{n-k}$ will produce a zero output

- What do we mean by saying that dimension of rowspace is k? If $x \in \mathbb{R}^n$ then why is the dimension not n.
- It means that of all the possible vectors in $\mathbb{R}^n$ only a subspace of vectors lying in $\mathbb{R}^k$ can act as inputs to Ax and produce a non-zero output. The remaining vectors in $\mathbb{R}^{n-k}$ will produce a zero output
- Hence we need only k dimensions to represent x

$$x = \sum_{i=1}^{k} \alpha_i v_i$$

- Let's look at a way of writing this as a matrix operation

$$Av_1 = \sigma_1 u_1, Av_2 = \sigma_2 u_2, \cdots, Av_k = \sigma_k u_k$$

$$A_{m\times n} V_{n\times k} = U_{m\times k} \underbrace{\Sigma_{k\times k}}_{\text{diagonal matrix}}$$

- Let's look at a way of writing this as a matrix operation

$$Av_1 = \sigma_1 u_1, Av_2 = \sigma_2 u_2, \cdots, Av_k = \sigma_k u_k$$

$$A_{m \times n} V_{n \times k} = U_{m \times k} \underbrace{\Sigma_{k \times k}}_{\text{diagonal matrix}}$$

- If we have k orthogonal vectors $(V_{n \times k})$ then using Gram Schmidt orthogonalization, we can find $n - k$ more orthogonal vectors to complete the basis for $\mathbb{R}^n$ [We can do the same for U]

$$A_{m \times n} V_{n \times n} = U_{m \times m} \Sigma_{m \times n}$$

$$U^T A V = \Sigma \qquad [U^{-1} = U^T] \qquad A = U \Sigma V^T \qquad [V^{-1} = V^T]$$

- Let's look at a way of writing this as a matrix operation

$$Av_1 = \sigma_1 u_1, Av_2 = \sigma_2 u_2, \cdots, Av_k = \sigma_k u_k$$

$$A_{m\times n} V_{n\times k} = U_{m\times k} \underbrace{\Sigma_{k\times k}}_{\text{diagonal matrix}}$$

- If we have k orthogonal vectors ($V_{n\times k}$) then using Gram Schmidt orthogonalization, we can find $n-k$ more orthogonal vectors to complete the basis for $\mathbb{R}^n$ [We can do the same for U]

$$A_{m\times n} V_{n\times n} = U_{m\times m} \Sigma_{m\times n}$$

$$U^T A V = \Sigma \qquad [U^{-1} = U^T] \qquad A = U \Sigma V^T \qquad [V^{-1} = V^T]$$

- Σ is a diagonal matrix with only the first k diagonal elements as non-zero

- Let's look at a way of writing this as a matrix operation

$$Av_1 = \sigma_1 u_1, Av_2 = \sigma_2 u_2, \cdots, Av_k = \sigma_k u_k$$

$$A_{m\times n} V_{n\times k} = U_{m\times k} \underbrace{\Sigma_{k\times k}}_{\text{diagonal matrix}}$$

- If we have k orthogonal vectors $(V_{n\times k})$ then using Gram Schmidt orthogonalization, we can find $n-k$ more orthogonal vectors to complete the basis for $\mathbb{R}^n$ [We can do the same for U]

$$A_{m\times n} V_{n\times n} = U_{m\times m} \Sigma_{m\times n}$$

$$U^T AV = \Sigma \qquad [U^{-1} = U^T] \qquad A = U\Sigma V^T \qquad [V^{-1} = V^T]$$

- Σ is a diagonal matrix with only the first k diagonal elements as non-zero
- Now the question is how do we find V, U and Σ

- Suppose V, U and Σ exist, then

- Suppose V, U and Σ exist, then

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T)$$

- Suppose V, U and Σ exist, then

$$
\begin{aligned}
A^T A &= (U\Sigma V^T)^T (U\Sigma V^T) \\
&= V\Sigma^T U^T U\Sigma V^T
\end{aligned}
$$

- Suppose V, U and Σ exist, then

$$\begin{aligned}A^TA &= (U\Sigma V^T)^T(U\Sigma V^T)\\ &= V\Sigma^T U^T U\Sigma V^T\\ A^TA &= V\Sigma^2 V^T\end{aligned}$$

- Suppose V, U and Σ exist, then

$$\begin{aligned} A^TA &= (U\Sigma V^T)^T(U\Sigma V^T) \\ &= V\Sigma^T U^T U\Sigma V^T \\ A^TA &= V\Sigma^2 V^T \end{aligned}$$

- What does this look like?

- Suppose V, U and Σ exist, then

$$\begin{aligned} A^T A &= (U\Sigma V^T)^T (U\Sigma V^T) \\ &= V\Sigma^T U^T U\Sigma V^T \\ A^T A &= V\Sigma^2 V^T \end{aligned}$$

- What does this look like? Eigen Value decomposition of $A^T A$

- Suppose V, U and Σ exist, then

$$\begin{aligned} A^T A &= (U\Sigma V^T)^T (U\Sigma V^T) \\ &= V\Sigma^T U^T U\Sigma V^T \\ A^T A &= V\Sigma^2 V^T \end{aligned}$$

- What does this look like? Eigen Value decomposition of $A^T A$
- Similarly we can show that

$$AA^T = U\Sigma^2 U^T$$

- Suppose V, U and Σ exist, then

$$\begin{aligned} A^TA &= (U\Sigma V^T)^T(U\Sigma V^T) \\ &= V\Sigma^T U^T U\Sigma V^T \\ A^TA &= V\Sigma^2 V^T \end{aligned}$$

- What does this look like? Eigen Value decomposition of A^TA
- Similarly we can show that

$$AA^T = U\Sigma^2 U^T$$

- Thus U and V are the eigen vectors of AA^T and A^TA respectively and $\Sigma^2 = \Lambda$ where Λ is the diagonal matrix containing eigen values of A^TA

$$
\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{m\times n} = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ u_1 & \cdots & u_k \\ \downarrow & \cdots & \downarrow \end{bmatrix}_{m\times k} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix}_{k\times k} \begin{bmatrix} \leftarrow & v_1 & \rightarrow \\ & \vdots & \\ \leftarrow & v_k & \rightarrow \end{bmatrix}_{k\times n}
$$

$$
= \sum_{i=1}^{k} \sigma_i u_i v_i^T
$$

$$
\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{m\times n} = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ & & \\ u_1 & \cdots & u_k \\ \downarrow & \cdots & \downarrow \end{bmatrix}_{m\times k} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix}_{k\times k} \begin{bmatrix} \leftarrow & v_1 & \rightarrow \\ & \vdots & \\ \leftarrow & v_k & \rightarrow \end{bmatrix}_{k\times n}
$$

$$
= \sum_{i=1}^{k} \sigma_i u_i v_i^T
$$

Theorem:

$\sigma_1 u_1 v_1^T$ is the best rank-1 approximation of the matrix A. $\sum_{i=1}^{2} \sigma_i u_i v_i^T$ is the best rank-2 approximation of matrix A. In general, $\sum_{i=1}^{k} \sigma_i u_i v_i^T$ is the best rank-k approximation of matrix A. In other words, the solution to

$$
\min \|A - B\|_F^2 \quad \text{is given by :}
$$

$$
B = U_{.,k} \Sigma_{k,k} V_{k,.}^T \quad \text{(minimizes reconstruction error of A)}
$$

$$\sigma_i = \sqrt{\lambda_i} = \text{singular value of A}$$

$$\sigma_i = \sqrt{\lambda_i} = \text{singular value of A}$$

$$U = \text{left singular matrix of A}$$

$$\sigma_i = \sqrt{\lambda_i} = \text{singular value of A}$$
$$U = \text{left singular matrix of A}$$
$$V = \text{right singular matrix of A}$$