CS6015: Linear Algebra and Random Processes
Quiz - 2
Course Instructor : Prashanth L.A.
Date : Aug-29, 2019 **Duration** : 35 minutes

INSTRUCTIONS: Answers should be given with proper justification. Please use rough sheets for any calculations *if necessary.* Please **DO NOT** submit the rough sheets. Please DO NOT use pencil for writing the answers.

Assume standard data whenever you feel that the given data is insufficient. However, please do quote your assumptions explicitly.

1. True or False? Answer any five.

   Note: 2 *marks for the correct answer and* $-\frac{1}{2}$ *for the wrong answer.*

   (a) If vectors u and v are orthogonal, and P is a projection, then Pu and Pv are orthogonal.

   Solution: False.

   (b) There exists a matrix whose row space contains $(1, 0, -1)$, and whose null space contains $(0, 1, 1)$.

   Solution: False.

   (c) If the columns of a square matrix are orthonormal, then its rows are orthonormal as well.

   Solution: True.

   (d) For any subspace S, let $S^{\perp}$ denote its orthogonal complement. If U and W are subspaces of V and $U \subset W$, then $U^{\perp} \subset W^{\perp}$.

   Solution: False.

   (e) If $\|u\| = \|v\|$, then $u + v$ and $u - v$ are orthogonal.

   Solution: True.

   (f) The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -7x - 15y \\ 6xy \end{bmatrix}$ is a linear transformation.

   Solution: False.

2. Let $A = \begin{bmatrix} 1 & -6 \\ 3 & 6 \\ 4 & 8 \\ 5 & 0 \\ 7 & 8 \end{bmatrix}$.

   Answer the following: (3 + 3 + 4 marks)

   (a) Find an orthonormal basis for the column space of A.

   (b) Write A as QR, where Q has orthonormal columns and R is upper triangular.

   (c) Let $b = (-3, 7, 1, 0, 4)$. Find the least-squares solution to $Ax = b$.

Solution:

Normalizing the first column, we obtain $q_1 = \frac{1}{10}\begin{bmatrix}1\\3\\4\\5\\7\end{bmatrix}$. Let $c_2 = \begin{bmatrix}-6\\6\\8\\0\\8\end{bmatrix}$ denote the second column.

Then,

$$b_2 = c_2 - (q_1^\top c_2)q_1 = \begin{bmatrix}-6\\6\\8\\0\\8\end{bmatrix} - \begin{bmatrix}1\\3\\4\\5\\7\end{bmatrix} = \begin{bmatrix}-7\\3\\4\\-5\\1\end{bmatrix}$$

Now, $q_2 = b_2 / \|b_2\| = \frac{1}{10}\begin{bmatrix}-7\\3\\4\\-5\\1\end{bmatrix}$.

The QR-factorization is given y

$$A = QR = \frac{1}{10}\begin{bmatrix}1 & -7\\3 & 3\\4 & 4\\5 & -5\\7 & 1\end{bmatrix}\begin{bmatrix}10 & 10\\0 & 10\end{bmatrix}.$$

Finally, the least squares approximation is

$$Rx = Q^\top b, \text{ i.e.,}$$

$$\begin{bmatrix}10 & 10\\0 & 10\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \frac{1}{10}\begin{bmatrix}1 & 3 & 4 & 5 & 7\\-7 & 3 & 4 & -5 & 1\end{bmatrix}\begin{bmatrix}-3\\7\\1\\0\\4\end{bmatrix}$$

leading to

$$x_1 = 0, x_2 = \frac{1}{2}.$$