CS6015: Linear Algebra and Random Processes
Course Instructor : Prashanth L.A.
Quiz - 7: Solutions

1. True or False?

   (a) Let $\text{Cov}(X, Y)$ denote the covariance of X and Y. If $\text{Cov}(X, Y) = 0$, then X and Y are independent.

   Solution: False

   (b) If $\text{Var}(X) = 0$, then $\mathbb{P}(X = c) = 1$ for some $c \in \mathbb{R}$.

   Solution: True

   (c) There does not exist a r.v. X that satisfies $\mathbb{E}\left(\frac{1}{X}\right) = \frac{1}{\mathbb{E}(X)}$.

   Solution: False

   (d) If $\mathbb{E}(X) < \infty$ then $\text{Var}(X) < \infty$.

   Solution: False

   (e) If $\text{Var}(X) < \infty$ then $\mathbb{E}(X) < \infty$.

   Solution: True

   (f) Let $\rho(X, Y)$ denote the correlation coefficient of X and Y. Then, for any a, b, c, d with $a > 0, c > 0$,

$$\rho(aX + b, cY + d) = \rho(X, Y).$$

   Solution: True

2. Let X and Y be independent r.v.s taking values $1, 2, 3, 4$, each with probability $\frac{1}{4}$. Let $Z = \max(X, Y)$.

   Answer the following:

   (a) Write down the joint distribution of X and Z?

   Solution: The joint mass function of X and Z is given by

	$Z = 1$	$Z = 2$	$Z = 3$	$Z = 4$	f_X
$X = 1$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$
$X = 2$	0	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$
$X = 3$	0	0	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{1}{4}$
$X = 4$	0	0	0	$\frac{4}{16}$	$\frac{1}{4}$
f_Z	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$	

   Using the table above, it is straightforward (and tedious) to write down the joint distribution F of X and Z, though I meant to ask only the joint mass function and not F.

   (b) Find $\mathbb{E}(X)$ and $\mathbb{E}(Z)$.

Solution: $\mathbb{E}(X) = \frac{5}{2}$ and $\mathbb{E}(Z) = \frac{25}{8}$

(c) Find $\mathrm{Cov}(X, Z)$.

Solution: $\mathrm{Cov}(X, Z) = \frac{5}{8}$ can be inferred from the following table coupled with part (a):

	$Z - \mathbb{E}(Z) = -\frac{17}{8}$	$Z - \mathbb{E}(Z) = -\frac{9}{8}$	$Z - \mathbb{E}(Z) = -\frac{1}{8}$	$Z - \mathbb{E}(Z) = \frac{7}{8}$
$X - \mathbb{E}(X) = -\frac{3}{2}$	$\frac{51}{16}$	$\frac{27}{16}$	$\frac{3}{16}$	$-\frac{21}{16}$
$X - \mathbb{E}(X) = -\frac{1}{2}$	$-$	$\frac{9}{16}$	$\frac{1}{16}$	$-\frac{7}{16}$
$X - \mathbb{E}(X) = \frac{1}{2}$	$-$	$-$	$-\frac{1}{16}$	$\frac{7}{16}$
$X - \mathbb{E}(X) = \frac{3}{2}$	$-$	$-$	$-$	$\frac{21}{16}$

In the above, we tabulated $(X - \mathbb{E}(X))(Z - \mathbb{E}(Z))$ and $\mathrm{Cov}(X, Z)$ can be inferred by multplying the entry in the table above with the corresponding probability from part (a) and summing them up.