CS6015: Linear Algebra and Random Processes
Course Instructor : Prashanth L.A.
Quiz - 8: Solutions

1. True or False?

   (a) If X and Y are independent r.v.s, then $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$.

   Solution: True.

   (b) Let X be a r.v. with continuous distribution F. Then, for some $a \in \mathbb{R}$,

   $$\int_{-\infty}^{a} F(x)dx = \int_{a}^{\infty} (1 - F(x))dx.$$

   Solution: False in general. However, for r.v.s with finite mean the claim is true for $a = \mathbb{E}(X)$ and for this choice of a only.

   (c) Let X_1 and X_2 be independent and have the common geometric distribution with parameter p. Then,

   $$\mathbb{P}\left(X_1 = k \mid X_1 + X_2 = n\right) = \frac{1}{n-1}, \qquad k = 1, \ldots, n-1.$$

   Solution: True and this can be argued without resorting to calculating the conditional probability explicitly using pmfs of X_i.

   (d) X is memoryless if $\mathbb{P}\left(X > s + t \mid X > s\right) = \mathbb{P}\left(X > t\right)$. Exponential distribution is the only continuous distribution with this property.

   Solution: True. Any probability text will have the proof with high probability.

   (e) If f and g are density functions, then $\alpha f + (1 - \alpha) g$ is a density function for any $\alpha \in [0, 1]$.

   Solution: True.

   (f) Let X be a non-negative r.v. with density f. Then,

   $$\mathbb{E}(X^2) = \int_0^{\infty} x\mathbb{P}\left(X > x\right) dx.$$

   Solution: False, since $\mathbb{E}(X^2) = \int_0^{\infty} 2x\mathbb{P}\left(X > x\right) dx$.

2. Two balls are drawn from a bag containing four balls numbered 1, 2, 3 and 4. If at least one of the numbers drawn is greater than 2, you win 10 rupees and otherwise you lose the same amount. Let X be the total amount won or lost and Y be the first number drawn.

   Answer the following:

   (a) Find $\mathbb{E}\left(X \mid Y = 1\right)$.

Solution:

$$\begin{aligned}\mathbb{E}(X \mid Y=1) &= -10\,\mathbb{P}(X=-10 \mid Y=1) + 10\,\mathbb{P}(X=10 \mid Y=1)\\ &= -10\frac{\mathbb{P}(X=-10, Y=1)}{\mathbb{P}(Y=1)} + 10\frac{\mathbb{P}(X=10, Y=1)}{\mathbb{P}(Y=1)}\\ &= -10\frac{\frac{1}{4}\frac{1}{3}}{\frac{1}{4}} + 10\frac{\frac{1}{4}\frac{1}{3}2}{\frac{1}{4}}\\ &= \frac{10}{3}.\end{aligned}$$

(b) Find $\mathbb{E}(X \mid Y=3)$.

Solution: A straightforward calculation shows that $\mathbb{E}(X \mid Y=3) = 10$

(c) Find $\mathbb{E}(X \mid Y)$ and $\mathbb{E}(X)$.

Solution:

$$\mathbb{E}(X \mid Y)(\omega) = \begin{cases} \frac{10}{3}, & \text{if } \{Y(\omega)=1 \text{ or } Y(\omega)=2\},\\ 10, & \text{if } \{Y(\omega)=3 \text{ or } Y(\omega)=4\}.\end{cases}$$

Finally,

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(\mathbb{E}(X \mid Y))\\ &= \frac{10}{3}\,\mathbb{P}(Y=1) + \frac{10}{3}\,\mathbb{P}(Y=2) + 10\,\mathbb{P}(Y=3) + 10\,\mathbb{P}(Y=4)\\ &= \frac{20}{3}.\end{aligned}$$