Lecture-14

(I) Recall $P^{(n)} \to$ n-step t.p.m.

$P^{(n)}_{i,j} \to$ prob. of going from i to j in "n" steps

$$P^{(n)} = P^n$$

Q: Does $P^{(n)}$ converge as $n \to \infty$?

(II) Occupancy matrix $M^{(n)} = \sum_{r=0}^{n} P^r$

$M^{(n)}_{i,j} \to$ # visits to j starting in i up to time n

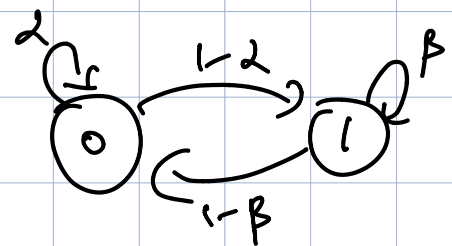
$M^{(n)} \to$ row sums are "$n+1$"

Q: Does $\frac{M^{(n)}}{n+1}$ converge as $n \to \infty$?

EXAMPLES

① Two-state DTMC:

$\alpha + \beta < 2$



By an induction argument, it can be shown that

$$P^n = \frac{1}{2-\alpha-\beta}\begin{bmatrix} 1-\beta & 1-\alpha \\ 1-\beta & 1-\alpha \end{bmatrix} + \frac{(\alpha+\beta-1)^n}{2-\alpha-\beta}\begin{bmatrix} 1-\alpha & \alpha-1 \\ \beta-1 & 1-\beta \end{bmatrix}, \ \forall n$$

H.W. Check case $n=1$ $P^1 = P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$

induction hypothesis for $k=1 \ldots n$ & prove for "$n+1$" using $P^{n+1} = P^n P$

$$\lim_{n\to\infty} P^n = \frac{1}{2-\alpha-\beta}\begin{bmatrix} 1-\beta & 1-\alpha \\ 1-\beta & 1-\alpha \end{bmatrix}$$

Another induction argument gives

$$M^{(n)} = \frac{n+1}{2-\alpha-\beta}\begin{bmatrix} 1-\beta & 1-\alpha \\ 1-\beta & 1-\alpha \end{bmatrix} + \frac{1-(\alpha+\beta-1)^{n+1}}{(2-\alpha-\beta)^2}\begin{bmatrix} 1-\alpha & \alpha-1 \\ \beta-1 & 1-\beta \end{bmatrix}$$

$$\lim_{n\to\infty} \frac{M^{(n)}}{n+1} = \frac{1}{2-\alpha-\beta}\begin{bmatrix} 1-\beta & 1-\alpha \\ 1-\beta & 1-\alpha \end{bmatrix}$$

P^n & $\frac{M^{(n)}}{n+1}$ converged to the same limit.

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(2) Three-state DTMC

$$P = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix} \qquad 0<p<1, \quad q=1-p$$

$$P^{(2n)} = \begin{bmatrix} q & 0 & p \\ 0 & 1 & 0 \\ q & 0 & p \end{bmatrix}, \quad n \geq 1$$

$$P^{(2n+1)} = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}, \quad n \geq 0$$

" $\lim_{n\to\infty} P^{(n)}$ does not exist "

$$M^{(2n)} = \begin{bmatrix} 1+nq & n & np \\ nq & 1+n & np \\ nq & n & 1+np \end{bmatrix} \qquad \frac{M^{(2n)}}{2n+1} \xrightarrow[n\to\infty]{} \begin{bmatrix} \frac{q}{2} & \frac{1}{2} & \frac{p}{2} \\ \frac{q}{2} & \frac{1}{2} & \frac{p}{2} \\ \frac{q}{2} & \frac{1}{2} & \frac{p}{2} \end{bmatrix}$$

$$M^{(2n+1)} = \begin{bmatrix} 1+nq & 1+n & np \\ (n+1)q & 1+n & (n+1)p \\ nq & 1+n & 1+np \end{bmatrix} \qquad \frac{M^{(2n+1)}}{2n+2} \xrightarrow[n\to\infty]{} \begin{bmatrix} q/2 & 1/2 & p/2 \\ q/2 & 1/2 & p/2 \\ q/2 & 1/2 & p/2 \end{bmatrix}$$

$$\lim_{n\to\infty} \frac{M^{(n)}}{n+1} = \begin{bmatrix} \frac{q}{2} & \frac{1}{2} & \frac{p}{2} \\ \frac{q}{2} & \frac{1}{2} & \frac{p}{2} \\ \frac{q}{2} & \frac{1}{2} & \frac{p}{2} \end{bmatrix}$$

---

③ Random walk

$$P_{i,i+1} = p, \quad P_{i,i-1} = q \; (=1-p), \quad \forall i$$

From the past, we have

$$P_{0,0}^{(2n)} = \frac{(2n)!}{n!\, n!} p^n q^n, \quad \forall n$$

Stirling's approximation: $n! \sim \sqrt{2\pi}\, n^{\frac{n+1}{2}} e^{-n}$

$$P_{0,0}^{(2n)} \sim \frac{\sqrt{2\pi}\,(2n)^{\frac{2n+1}{2}}\, \cancel{e^{-2n}}}{\left(\sqrt{2\pi}\, n^{\frac{n+1}{2}}\, \cancel{e^{-n}}\right)^2} p^n q^n$$

$$= \frac{(4pq)^n}{\sqrt{\pi n}} \qquad pq = p(1-p) \le \frac{1}{4}$$

$$4pq \le 1$$

$$\le \frac{1}{\sqrt{\pi n}}$$

$$P_{0,0}^{(2n)} \xrightarrow[n\to\infty]{} 0 \quad \& \quad P_{0,0}^{(2n+1)} = 0$$

$$P_{0,0}^{(n)} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

Similarly, $\lim\limits_{n\to\infty} \left[P^{(n)}\right]_{i,j} = 0$ (Check this)

It can be shown using "tedious" calculations that

$$\lim_{n\to\infty} \frac{\left[M^{(n)}\right]_{i,j}}{n+1} = 0$$

<u>Remark:</u> $\sum\limits_{j=-\infty}^{\infty} P_{i,j}^{(n)} = 1 \Rightarrow \lim\limits_{n\to\infty} \sum\limits_{j} P_{i,j}^{(n)} = 1$

But, $\sum\limits_{j=-\infty}^{\infty} \lim\limits_{n\to\infty} P_{i,j}^{(n)} = 0$

So, cannot interchange an (infinite) sum & limit.

Classification of states

"Accessibility"

A state j is said to be accessible from a state i if $\exists n \geq 0$ s.t. $P_{i,j}^{(n)} > 0$.

If j is accessible from i, we write $i \to j$

$i \to j \Rightarrow \exists$ a directed path from i to j in the transition diagram

Communication: States i and j communicate if $i \to j$ and $j \to i$

Denote communication by "$\leftrightarrow$"

Claim:

(i) $i \leftrightarrow i$ (reflexive)

(ii) If $i \leftrightarrow j$ then $j \leftrightarrow i$ (symmetric)

(iii) If $i \leftrightarrow j$, $j \leftrightarrow k$, then $i \leftrightarrow k$ (transitive)

Pf: (i) & (ii) hold by definition

For (iii) $i \to j$, $j \to k$

$\Rightarrow \exists n, m$ s.t. $P_{ij}^{(n)} > 0$ & $P_{j,k}^{(m)} > 0$

$$\Rightarrow \quad p^{(n+m)}_{i,k} \underset{\text{Chapman Kolmogorov}}{=} \sum_{r\in S} p^{(n)}_{i,r}\, p^{(m)}_{r,k}$$

$$\geq p^{(n)}_{i,j}\, p^{(m)}_{j,k} > 0$$

$$\Rightarrow \quad i \to k$$

Similarly, using $k \to j$ & $j \to i$, we have $k \to i$

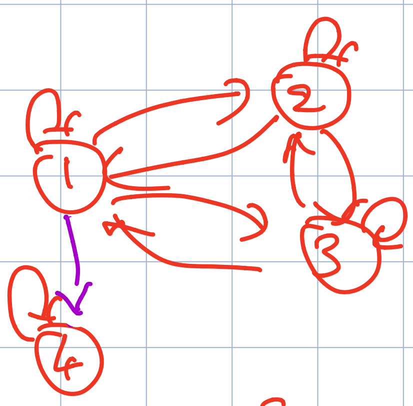
So, $i \leftrightarrow k$

Communicating class:

A set $C \subset S$ is a communicating class if

(i) $i \in C,\ j \in C \Rightarrow i \leftrightarrow j$

(ii) $i \in C,\ i \leftrightarrow j \Rightarrow j \in C$ ← makes C maximal



$1 \leftrightarrow 2 \quad 2 \leftrightarrow 3$

$\{1,2\}$ is not a communicating class

but $\{1,2,3\}$ is.

$S = \{1,2,3\} \cup \{4\}$

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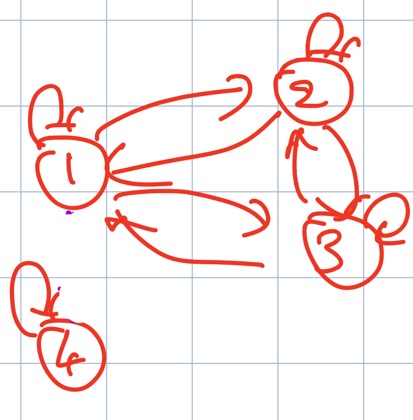
Closed communicating class

A communicating class C is closed if for any $i \in C$ & $j \notin C$, we have $\underbrace{i \not\to j}$

j is not accessible from i



$\{1, 2, 3\}$ is a closed communicating class

If $X_n \in C$ for some n, & C is closed, then
$X_m \in C \quad \forall\, m \geq n$.

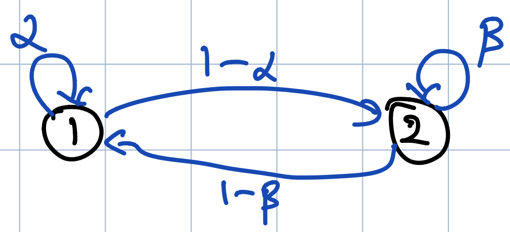
We can partition the state space as

$$S = \underbrace{C_1 \cup C_2 \cup \cdots \cup C_k}_{\text{disjoint closed communicating classes}} \cup \underbrace{T}_{\text{left-over states (could include communicating classes that aren't closed)}}$$

Irreducibility: If the state space S is a single closed communicating class, then the DTMC is said to be irreducible.

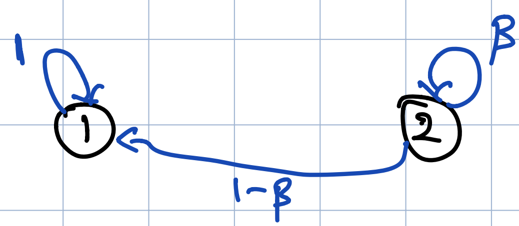
Else it is reducible.

Example:



If $0 \le \alpha, \beta < 1$, then $\{1,2\}$ is a closed communicating class & the DTMC is irreducible.

Suppose $\alpha = 1$ & $0 < \beta < 1$



$\{1\}$ is a closed communicating class

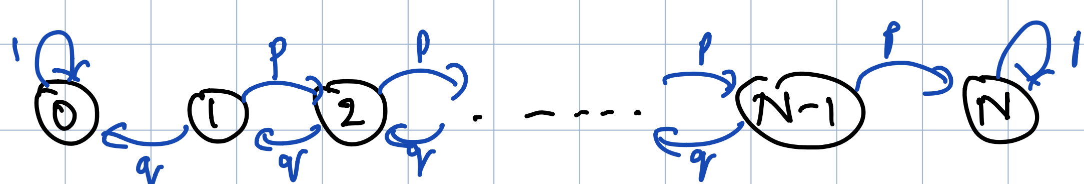
$\{2\}$ is not closed

Partition of state space $= \underbrace{\{1\}}_{C_1} \cup \underbrace{\{2\}}_{T}$

Case $\alpha = \beta = 1$: Partition: $\underbrace{\{1\}}_{C_1} \cup \underbrace{\{2\}}_{C_2}$ $\quad T = \phi$

Example: Random walk

$P_{0,0} = P_{N,N} = 1, \quad P_{i,i+1} = p, \quad P_{i,i-1} = q, \quad 1 \le i \le N-1$



$0 < p, q < 1 \quad , \quad p+q=1$

$C_1 = \{0\} \quad , \quad C_2 = \{N\}, \quad T = \underbrace{\{1, 2, \ldots, N-1\}}_{\text{communicating class that ain't closed}}$

Recurrence & transience:

$\tilde{T}_i = \min\{n > 0 \mid X_n = i\}, \ i \in S$

$\tilde{u}_i = P(\tilde{T}_i < \infty \mid X_0 = i)$ → Prob of returning to state i

$\tilde{m}_i = E(\tilde{T}_i \mid X_0 = i)$ → expected # steps taken before returning

Def: A state i is $\begin{cases} \text{recurrent if } \tilde{u}_i = 1 \\ \text{transient if } \tilde{u}_i < 1 \end{cases}$

Q: If $\underbrace{\tilde{u}_i < 1}_{\text{state is transient}}$, then $\underbrace{\tilde{m}_i = \infty}_{\text{mean \# steps for return}}$.

Q: If $\tilde{u}_i = 1$, then is it true that $\tilde{m}_i < \infty$?

A: No

Example: Success runs

$P_{i,i+1} = \frac{i+1}{i+2}, \quad P_{i,0} = \frac{1}{i+2}, \quad i \geq 0$

$$P(\tilde{T}_0 = n \mid X_0 = 0) = P(X_1 = 1, X_2 = 2, \ldots, X_{n-1} = n-1, X_n = 0 \mid X_0 = 0)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \cdots \times \frac{n-1}{n} \times \frac{1}{n+1}$$

$$P(\tilde{T}_0 = n \mid X_0 = 0) = \frac{1}{n(n+1)}$$

$$\tilde{u}_0 = P(\tilde{T}_0 < \infty \mid X_0 = 0)$$

$$= \sum_{n=1}^{\infty} P(\tilde{T}_0 = n \mid X_0 = 0)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Partial sum $Z_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$$= 1 - \frac{1}{n+1}$$

$Z_n \to 1$ as $n \to \infty$

So, $\tilde{u}_0 = 1$ i.e., "w.p. 1 I return to state 0"

0 is recurrent

Mean return time $\tilde{m}_0 = \sum_{n=1}^{\infty} n \, P(\tilde{T}_0 = n \mid X_0 = 0)$

$$= \sum_{n=1}^{\infty} n \times \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$$

So, $\tilde{u}_0 = 1$ but $\tilde{m}_0 = \infty$

Def: A recurrent state i is

positive recurrent if $\tilde{m}_i < \infty$

null-recurrent if $\tilde{m}_i = \infty$

Example: If i is an absorbing state, then

$$P(X_1 = i \mid X_0 = i) = 1$$

$$\tilde{u}_i = 1, \quad \tilde{m}_i = 1$$

Thus, i is positive recurrent.

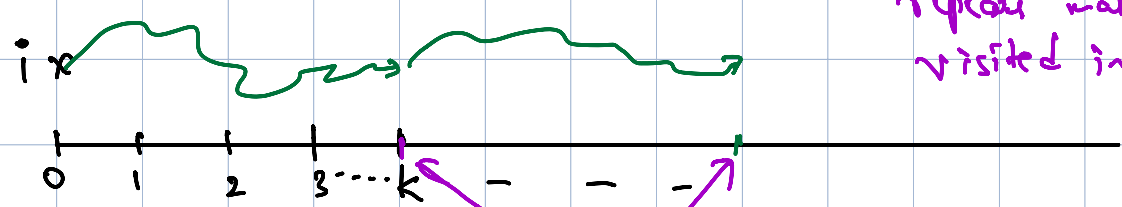
Recall $V_i^{(n)} \to$ # visits to state i over times $\{0, 1, \ldots, n\}$

Let $V_i \to$ # visits to state i over $\{0, 1, 2, \ldots\}$

If state i is recurrent, then

$$P(V_i = \infty \mid X_0 = i) = 1$$

Why? $X_0 = i$ & w.p. 1 I return to i & process repeats make "i" visited infinitely often



times where DTMC visits i

If state i is transient, then

$$P(V_i = k \mid X_0 = i) = \tilde{u}_i^{\,k-1}(1-\tilde{u}_i) \rightarrow \text{Geometric}$$

$\tilde{u}_i \rightarrow$ Prob. of returning to i

$1-\tilde{u}_i \rightarrow$ prob. of not returning to i

Criterion for recurrence/transience

$$E(V_i \mid X_0 = i) = \begin{cases} \infty & \text{if } \tilde{u}_i = 1 \\ \dfrac{1}{1-\tilde{u}_i} < \infty & \text{if } \tilde{u}_i < 1 \end{cases}$$

From the past, $E(V_i^{(n)} \mid X_0 = i) = M_{i,i}^{(n)} = \sum_{m=0}^{n} P_{i,i}^{(m)}$

In the limit, we have $E(V_i \mid X_0 = i) = \sum_{m=0}^{\infty} P_{i,i}^{(m)}$

So, $E(V_i \mid X_0 = i) = \sum_{m=0}^{\infty} P_{i,i}^{(m)}$

$= \infty$ if i is recurrent

$< \infty$ if i is transient

Claim: State i is $\begin{cases} \text{recurrent} & \text{if } \sum_{m=0}^{\infty} P_{i,i}^{(m)} = \infty \\ \text{transient} & \text{if } \sum_{m=0}^{\infty} P_{i,i}^{(m)} < \infty \end{cases}$

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The claim above gives a criterion for distinguishing between recurrent & transient states.

Next, we derive a criterion to distinguish between positive & null-recurrent states.

$\lim_{n\to\infty} \frac{1}{n+1} M_{i,i}^{(n)}$ → long run avg of #visits to i

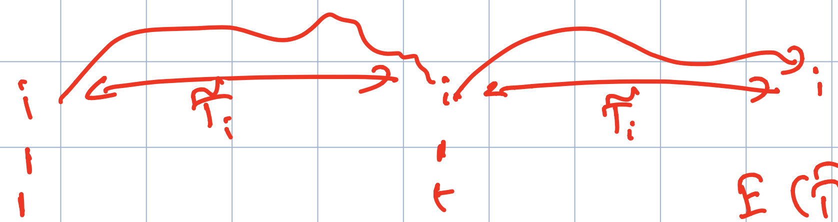
$\tilde{m}_i$ → mean time between visits to i

So, $\frac{1}{\tilde{m}_i}$ → # visits / unit time.

For a positive recurrent state, $\tilde{m}_i$ is finite
& for a null recurrent state, $\tilde{m}_i = \infty$.

Roughly, $\lim_{n\to\infty} \frac{1}{n+1} M_{i,i}^{(n)} \approx \frac{1}{\tilde{m}_i}$ → $= 0$ null-recurrent "i"
→ > 0 positive-recurrent "i"

intuitive justification



$E(\tilde{T}_i \mid X_0 = i) = \tilde{m}_i$

Claim without proof:

A recurrent state i is positive recurrent if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} P_{i,i}^{(m)} > 0$$

& null recurrent if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} P_{i,i}^{(m)} = 0$$

Recurrence & transience are (communicating) class properties

Claim:

(1) i is recurrent, $i \leftrightarrow j \Rightarrow j$ is recurrent

Pf: $i \leftrightarrow j \Rightarrow \exists\, n, m$ s.t. $P_{i,j}^{(n)} > 0$ & $P_{j,i}^{(m)} > 0$

$$P_{jj}^{(r+n+m)} \geq P_{ji}^{(m)} P_{ii}^{(r)} P_{ij}^{(n)}$$

(One way of getting from j to j is to go from j to i & then i to j)

$$\sum_{r=0}^{\infty} p_{jj}^{(r+n+m)} = \sum_{r=n+m}^{\infty} p_{jj}^{(r)}$$

$$\leq \sum_{r=0}^{\infty} p_{jj}^{(r)}$$

So, we have

$$\sum_{r=0}^{\infty} p_{jj}^{(r)} \geq \sum_{r=0}^{\infty} p_{jj}^{(r+n+m)}$$

$$\geq \sum_{r=0}^{\infty} p_{ji}^{(m)} p_{ii}^{(r)} p_{ij}^{(n)}$$

$$= p_{ji}^{(m)} p_{ij}^{(n)} \sum_{r=0}^{\infty} p_{ii}^{(r)}$$

$$= \infty$$

So, i recurrent & $i \leftrightarrow j \Rightarrow j$ is recurrent.

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(2) i transient, $i \leftrightarrow j \Rightarrow j$ is transient.

Suppose i is transient, $i \leftrightarrow j$ & j is recurrent

Since $j \leftrightarrow i$, j is recurrent, from part (1) above,

i is recurrent, a contradiction

Next, we will show positive/null recurrence are class properties

Claim: ① $i \leftrightarrow j$, i is positive recurrent
$\Rightarrow$ j is positive recurrent

Pf: $i \leftrightarrow j \Rightarrow \exists\, n, m$ s.t. $p_{i,j}^{(n)} > 0$ & $p_{j,i}^{(m)} > 0$

$$p_{jj}^{(r+n+m)} \geq p_{ji}^{(m)} p_{ii}^{(r)} p_{ij}^{(n)}$$

$$\lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{\infty} p_{j,j}^{(r)}$$

$$= \lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{\infty} p_{j,j}^{(r+n+m)}$$

The equality above holds because

$$\lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{\infty} p_{jj}^{(r)} = \underbrace{\lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{n+m-1} p_{jj}^{(r)}}_{\to 0} + \lim_{k\to\infty} \frac{1}{k+1} \sum_{r=n+m}^{\infty} p_{j,j}^{(r)}$$

$$= \lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{\infty} p_{j,j}^{(r+n+m)}$$

Hence, we have

$$\lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{\infty} p_{j,j}^{(r)} = \lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{\infty} p_{j,j}^{(r+n+m)}$$

$$\geq \lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{\infty} p_{ji}^{(m)} p_{ii}^{(r)} p_{ij}^{(n)}$$

$$= p_{ji}^{(m)} p_{ij}^{(n)} \Big[\underbrace{\lim_{k\to\infty} \frac{1}{k+1} \sum_{r=0}^{\infty} p_{ii}^{(r)}}_{} \Big]$$

$\hookrightarrow$ i is recurrent & hence this lim > 0

> 0

② $i \leftrightarrow j$, i is null recurrent
$\Rightarrow$ j is null recurrent

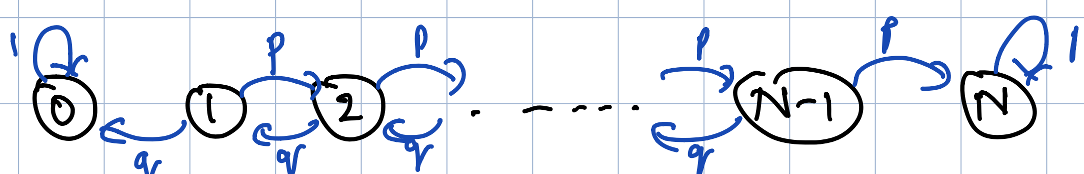
Pf: Follows using part ①.

(I) A communicating class is called
(i) "transient" if all its states are transient
(ii) "positive recurrent" if all its states are positive recurrent
(iii) "null recurrent" if all its states are null recurrent

(II) An irreducible DTMC is positive (null recurrent, transient) if all its states are positive (null recurrent, transient).

Example: Random walk

$$P_{0,0} = P_{N,N} = 1, \quad P_{i,i+1} = p, \quad P_{i,i-1} = q, \quad 1 \le i \le N-1$$

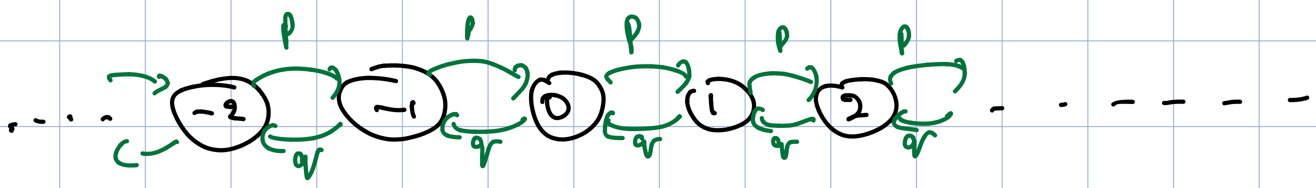


$0 < p, q < 1$, $p + q = 1$

States $0, N$: positive recurrent

$T = \{1, 2, \cdots N-1\}$: transient (argue this)

Example: Another random walk



$p>0, q>0, p+q=1$

The DTMC is irreducible.

Determining transience / recurrent of state 0 is sufficient

$$P_{0,0}^{(2n)} \sim \frac{(4pq)^n}{\sqrt{\pi n}} \quad , \quad P_{0,0}^{(2n+1)} = 0$$

$$\sum_{n=0}^{\infty} P_{0,0}^{(n)} = \sum_{n=0}^{\infty} \frac{(4pq)^n}{\sqrt{\pi n}}$$

Case 1: $p \neq q$, $4pq = c < 1$

$$\sum_{n=0}^{\infty} P_{0,0}^{(n)} = \sum_{n=0}^{\infty} \frac{(c)^n}{\sqrt{\pi n}} \rightarrow \text{converges}$$

$\underbrace{\sum \frac{\left(\frac{1}{2}\right)^n}{\sqrt{n}}}_{\text{converges}}$

$\underbrace{\sum \frac{1}{\sqrt{n}}}_{\text{diverges}}$

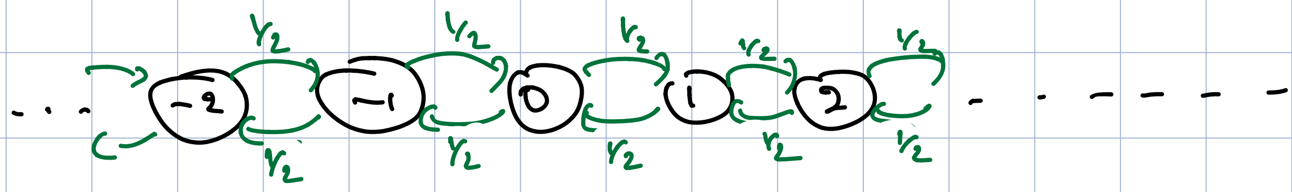
So, $\sum_{n=0}^{\infty} P_{0,0}^{(n)} < \infty \Rightarrow 0$ is transient.

Case 2: $p = q$, then $\sum_{n=0}^{\infty} P_{0,0}^{(n)} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$

$\Rightarrow 0$ is recurrent

In this case, the DTMC is null recurrent & this is argued as follows:

$m_i \to$ expected time to 0 from $i \neq 0$



$m_1 = m_{-1} = m$ (say)

First-step analysis: $\tilde{m}_0 = 1 + \frac{1}{2} m_1 + \frac{1}{2} m_{-1}$

$\tilde{m}_0 = 1 + m$ — (*)

Similarly, $m_1 = 1 + \frac{1}{2} m_2$ — (**)

Time to get to 0 from 2
= Time to get to 1 from 2 +
Time to get to 0 from 1

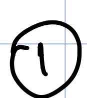
So, $m_2 = 2m_1 = 2m$

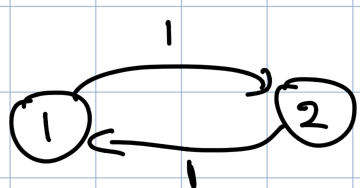
Using (*) $m = 1 + m \Rightarrow m = \infty$

So, $\tilde{m}_0 = \infty \Rightarrow$ 0 is null recurrent

Lecture-18

Periodicity:





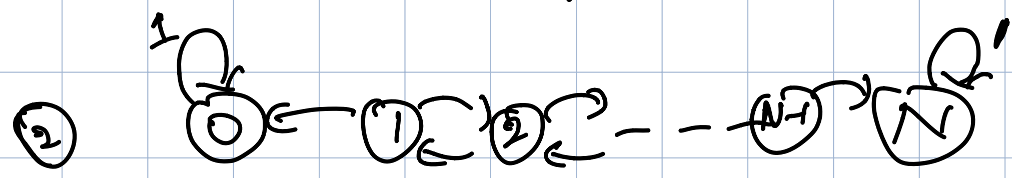
$X_0 = 1$

Visit instants of state 1
$= \{0, 2, 4, \ldots\}$

Period of state $1 = 2$

Visit instants of state $2 = \{1, 3, \ldots\}$

Period of state $2 = 2$



Period of $0 = 1 =$ Period of N

absorbing states

Def: Recall $\tilde{T}_i = \min\{n > 0 \mid X_n = i\}, \; i \in S$

Let "i" be a recurrent state. Let "d" be the largest positive integer such that

$$\sum_{k=1}^{\infty} P(\tilde{T}_i = kd) = 1.$$

If $d = 1$, then state i is aperiodic.

If $d > 1$, then state i is periodic with period d.

Equivalently, if "i" is a recurrent state with period d, then $p_{i,i}^{(n)} = 0$ $\forall n$ that are not positive integer multiples of d.

Periodicity is a class property:

Claim: If $i \leftrightarrow j$, then i and j have the same period.

Pf: Since $i \leftrightarrow j$, $\exists\, n, m$ s.t. $P_{i,j}^{(n)} > 0$, $P_{j,i}^{(m)} > 0$

Also, $$P_{i,i}^{(r+n+m)} \geq P_{i,j}^{(n)} P_{j,j}^{(r)} P_{j,i}^{(m)} \quad \text{— } (*)$$

Suppose "i" has a period "d".

With $r=0$ in $(*)$, we obtain

$$P_{i,i}^{(n+m)} \geq P_{i,j}^{(n)} P_{j,i}^{(m)} > 0$$

$(n+m)$ is a multiple of d.

Suppose r is not divisible by d.

$$P_{i,i}^{(r+n+m)} = 0 \implies P_{i,j}^{(n)} P_{j,j}^{(r)} P_{j,i}^{(m)} = 0$$

$$\implies P_{j,j}^{(r)} = 0$$

So, $P_{j,j}^{(r)} = 0$ for r that isn't divisible by d.

$\implies$ Period of j is an integer multiple of d, say it is kd for some $k \geq 1$.

Suppose j's period is $2d$

Then, $j \leftrightarrow i$, so i's period is an integer multiple of $\underline{2d}$ leading to a contradiction.

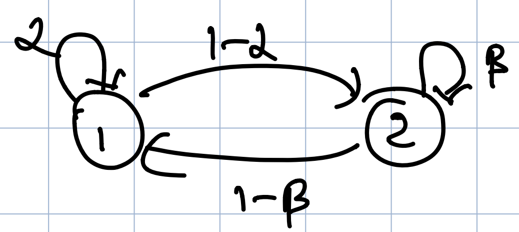
So, j's period is d.

Now, swapping roles of i & j, we can argue that i's period is an integer multiple of that of j, leading to "i & j have the same period".

■

So, can attach a "period" to a communicating class, or even a irreducible DTMC.

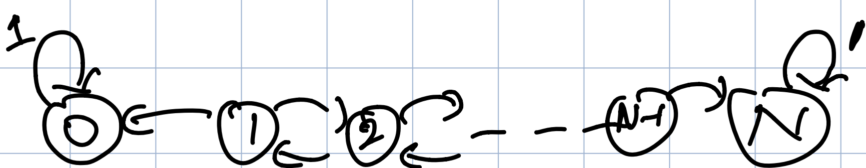
Example: (1)



Case $0<\alpha, \beta<1$: aperiodic

Case $\alpha=\beta=0$: periodic with $d=2$, $P=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

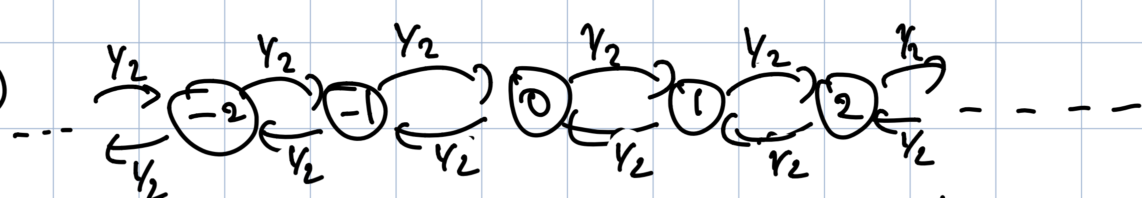
(2):



Period of $0 = 1 =$ Period of N

absorbing states

(3)



State 0 has period 2 & since DTMC is irreducible, all states have period 2.

Determining recurrence in a finite state DTMC

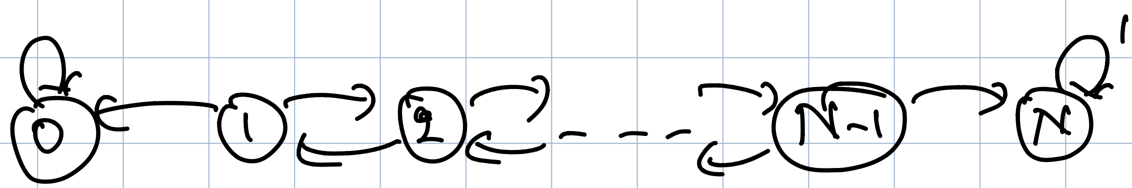
We will establish that

(I) all states in a finite closed communicating class are positive recurrent

(II) all states in a non-closed communicating class are transient

From I & II: All communicating classes of a finite DTMC are finite, which implies that they are either positive recurrent or transient.

Example:



$$\underbrace{\{0\}, \{N\}}_{\substack{\text{closed} \\ \Downarrow \\ \text{positive recurrent}}}, \quad \underbrace{\{1, \ldots, N-1\}}_{\substack{\text{not-closed.} \\ \Downarrow \\ \text{transient}}}$$

There are no "null-recurrent" communicating classes in a finite DTMC.

Proof of (I):

Let C be a finite closed communicating class.

Then, see next page

$$\forall i \in C, \quad 1 = P(X_k \in C \mid X_0 = i) = \sum_{j \in C} p_{i,j}^{(k)}$$

$$\frac{1}{n+1} \sum_{k=0}^{n} \sum_{j \in C} p_{i,j}^{(k)} = 1$$

Take limits, $$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j \in C} p_{i,j}^{(k)} = 1$$

Since C is finite, $$\sum_{j \in C} \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{i,j}^{(k)} = 1$$

There has to be at least one $j \in C$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{i,j}^{(k)} > 0 \quad \text{—} (*)$$

Since C is a communicating class, $\exists r$ s.t. $p_{j,i}^{(r)} > 0$

$$p_{j,j}^{(k+r)} \geq p_{j,i}^{(r)} \, p_{i,j}^{(k)} \quad \text{—} (**)$$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{j,j}^{(k)} = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{j,j}^{(k+r)}$$

$$\geq p_{j,i}^{(r)} \left(\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{i,j}^{(k)} \right)$$

So, $$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{j,j}^{(k)} > 0$$

$\Rightarrow$ j is positive recurrent

$\Rightarrow$ C is positive recurrent $\blacksquare$

Proof of claim II:

Let C be a non-closed communicating class.
$\exists\, i \in C$, $j \notin C$, such that i is not accessible from j, and $P_{i,j} > 0$

$\Rightarrow$ If the DTMC visits j starting from i, it never returns to i

$$P(\tilde{T}_i = \infty \mid X_0 = i) \geq P_{i,j} > 0$$

Recall $\tilde{u}_i = P(\tilde{T}_i < \infty \mid X_0 = i)$

So, $1 - \tilde{u}_i = P(\tilde{T}_i = \infty \mid X_0 = i) > 0$

$\Rightarrow \tilde{u}_i < 1$

$\Rightarrow$ i is transient

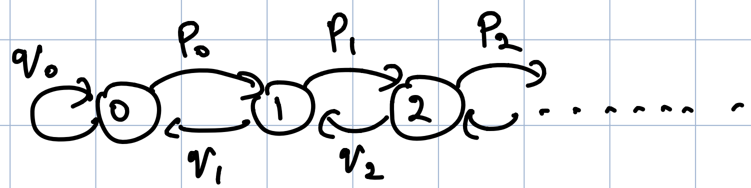
$\Rightarrow$ C is transient

$\blacksquare$

Lecture-19

Determining recurrence/transience in an infinite DTMC: An example

General simple random walk



Assume $p_i > 0,\ \forall i \geq 0$ & $q_i > 0\ \forall i \geq 1$.

DTMC is clearly irreducible

Using a first-step analysis as in the last chapter, one can infer the following:

$$v_1 = \begin{cases} > 0 & \text{if } \sum_{i=0}^{\infty} \alpha_i < \infty \\ 0 & \text{if } \sum_{i=0}^{\infty} \alpha_i = \infty, \end{cases}$$

where v_1 is the probability of not visiting 0 starting in 1,

and $\alpha_i = \dfrac{q_1 q_2 \cdots q_i}{p_1 p_2 \cdots p_i}$, $i \geq 1$, $\alpha_0 = 1$.

State 0 is $\begin{cases} \text{recurrent} & \text{if } \sum_{i=0}^{\infty} \alpha_i = \infty \\ \text{transient} & \text{if } \sum_{i=0}^{\infty} \alpha_i < \infty \end{cases}$

Since DTMC is irreducible, a similar claim holds for all states.

Determining positive/null recurrence:

Suppose $\sum \alpha_i = \infty$. So DTMC is recurrent.

$$\tilde{m}_0 = 1 + p_0 m_1$$

m_1 is the expected # steps to get to 0 starting in 1

From the previous chapter, we have

$$m_1 = \sum_{j=1}^{\infty} \frac{1}{p_j \alpha_j}$$

Change of notation: $\rho_0 = 1$, $\rho_i = \dfrac{p_0 p_1 \cdots p_{i-1}}{q_1 \cdots q_i}$, $i \geq 1$

So, $\tilde{m}_0 = \sum_{i=0}^{\infty} \rho_i$

$$p_j \alpha_j = \frac{q_1 \cdots q_j}{p_1 \cdots p_{j-1}}$$

$$m_1 = \sum_{j=1}^{\infty} \frac{p_1 \cdots p_{j-1}}{q_1 \cdots q_j}$$

$$\tilde{m}_0 = 1 + p_0 m_1 = 1 + \sum_{j=1}^{\infty} \frac{p_0 p_1 \cdots p_{j-1}}{q_1 \cdots q_j}$$

$$= \rho_0 + \sum_{j=1}^{\infty} \rho_j$$

So, $\tilde{m}_0 = \sum_{i=0}^{\infty} \rho_i$

Claim: Assume $\sum \alpha_i = \infty$. Then,

the DTMC is $\begin{cases} \text{positive recurrent} & \text{when } \sum_{i=0}^{\infty} e_i < \infty \\ \text{null - recurrent} & \text{when } \sum_{i=0}^{\infty} e_i = \infty \end{cases}$

If $\sum \alpha_i < \infty$, then the DTMC is transient

Limiting behaviour of irreducible DTMCs:

"The transient case"

Claim: Let $\{X_n, n \geq 0\}$ be an irreducible transient DTMC.

Then,

$$\lim_{n \to \infty} P_{i,j}^{(n)} = 0 , \quad \forall i, j$$

Pf: Since the DTMC is transient, we have (see the claim on p.12)

$$\sum_{n=0}^{\infty} P_{j,j}^{(n)} < \infty ,$$

which implies $\lim_{n \to \infty} P_{j,j}^{(n)} = 0$ since $P_{j,j}^{(n)} \geq 0$

If $i=j$, then we have the main claim.

Suppose $i \neq j$. Irreducibility $\Rightarrow j \leftrightarrow i$

$\Rightarrow j \to i$

$\Rightarrow p_{j,i}^{(m)} > 0$ for some $m > 0$

$$\underbrace{p_{j,j}^{(n)}}_{\downarrow \text{ as } n\to\infty \ 0} \geq \underbrace{p_{j,i}^{(m)}}_{\downarrow \ >0} \ \underbrace{p_{i,j}^{(n-m)}}_{\downarrow \ ??} \qquad \forall n \geq m$$

Since $p_{j,j}^{(n)} \xrightarrow[n\to\infty]{} 0$ & $p_{j,i}^{(m)} > 0$, we obtain

$$p_{i,j}^{(n-m)} \longrightarrow 0 \quad \text{as } (n-m) \to \infty$$

$\blacksquare$

Coming next:

(I) $\lim_{n\to\infty} P(X_n = j) = 0 \quad \forall j$

(II) Suppose the state space is finite.
Then, $\exists$ at least one recurrent state.

(III) Suppose the state space is finite and the DTMC is irreducible.
Then, all states are recurrent.

Lecture - 20

Pf of (I): Recall we have an irreducible transient DTMC.

Let $a = (a_i)_{i\in S}$ be the initial distribution

$$P(X_n = j) = \sum_{i\in S} a_i \, P_{i,j}^{(n)}$$

$$\lim_{n\to\infty} P(X_n = j) = \lim_{n\to\infty} \sum_{i\in S} a_i \, P_{i,j}^{(n)}$$

Suppose $|S| < \infty$. Then, we can interchange the limit & sum

$$\lim_{n\to\infty} P(X_n = j) = \sum_{i\in S} a_i \left(\lim_{n\to\infty} P_{i,j}^{(n)} \right)$$

$$\lim_{n\to\infty} P(X_n = j) = 0$$

Suppose $|S| = \infty$. Then, we can use dominated convergence theorem (DCT) to justify the interchange of lim & sum.

A precise argument is given below.

Let F be a finite subset of S.

$$\sum_{i\in S} a_i P_{i,j}^{(n)} = \sum_{i\in F} a_i P_{i,j}^{(n)} + \sum_{i\notin F} a_i P_{i,j}^{(n)}$$

$$\leq \sum_{i\in F} a_i P_{i,j}^{(n)} + \sum_{i\notin F} a_i$$

$$\xrightarrow{n\to\infty} \sum_{i\notin F} a_i$$

$$\xrightarrow{F\uparrow S} 0$$

So, $\lim_{n\to\infty} P(X_n = j) = 0.$

(II) Suppose the state space is finite.

Then, $\exists$ at least one recurrent state.

Pf:
$$\sum_{j\in S} P_{i,j}^{(n)} = 1.$$

Assume every state is transient, i.e.,
$$\lim_{n\to\infty} P_{i,j}^{(n)} = 0$$

This leads to a contradiction. ∎

(III) Suppose the state space is finite and the DTMC is irreducible.

Then, all states are recurrent.

Pf: From part (II), we have at least one recurrent state. + DTMC is irreducible

$\Rightarrow$ All states are recurrent. ∎

Coming later: In a finite irreducible DTMC, all states are positive recurrent.

Stationary distributions
(aka steady-state / invariant distributions)

Ref: Sec 6.4 of Grimmett-Stirzaker's book
or Sec 12.9 of Grimmett-Welsh's book

<u>Def:</u> For a Markov chain with t.p.m. P, the vector $\pi = (\pi_i, i \in S)$ is called a stationary distribution if

(i) $\pi_i \geq 0 \ \forall i$, $\sum_i \pi_i = 1$ ← π is a distribution

(ii) $\pi = \pi P$ ← π is stationary

<u>Remark:</u> If the initial distribution is π, then what is the distribution of X_n: πP^n

$$= \pi P \cdot P^{n-1}$$
$$= \pi P^{n-1}$$
$$\vdots$$
dist^n of X_n $= \pi$

<u>Remark:</u> If π satisfies $\pi = \pi P$ but not condition (i) in def above, then π is a stationary measure.

Main result: Consider an irreducible DTMC.

(a) $\exists$ a stationary distribution π if and only if some state is positive recurrent.

(i.e., If DTMC is irreducible + transient/null-recurrent, then NO stationary distribution)

(b) If there exists a stationary distribution π, then (i) every state is positive recurrent

(ii) $\pi_i = \dfrac{1}{m_i} \quad \forall i \in S$, where

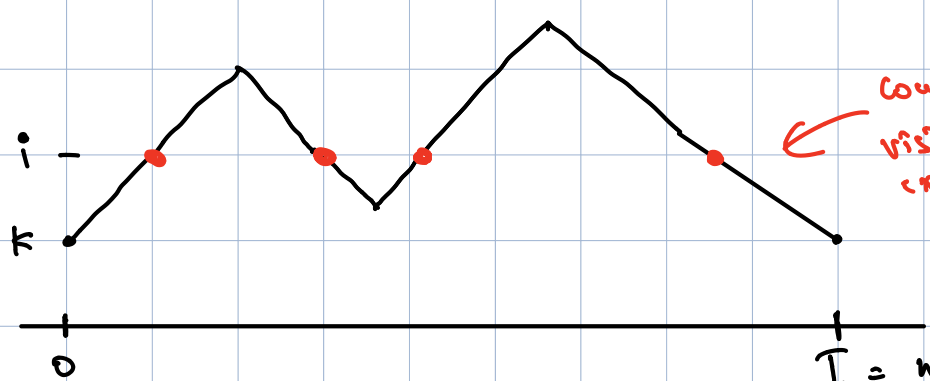
mean return time $m_i = E(T_i \mid X_0 = i)$, with $T_i = \min\{n \geq 1 \mid X_n = i\}$

[i.e., $\dfrac{1}{m_i} = \sum_j \dfrac{1}{m_j} p_{j,i}$, since $\pi_i = \sum_{j \in S} \pi_j p_{j,i} \iff \pi = \pi P$]

(iii) π is unique.

Construction of a stationary measure

We will make up a vector $e = (e_i, i \in S)$ that satisfies $e = eP$



$T_k = \min\{n \geq 1 \mid X_n = k\}$

count # visits to i between visits to k & set e_i to the expected value of this number.

$$W_i = \sum_{m=1}^{T_k} I(X_m = i) \quad \& \quad e_i = E(W_i \mid X_0 = k)$$

A bunch of claims: For an irreducible, recurrent DTMC, with e as constructed above, we have

(i) $e_k = 1$

(ii) $\sum_{i \in S} e_i = m_k$ ⟵ if DTMC is null-recurrent, $m_k = \infty$ else $m_k < \infty$ in the positive-recurrent case.

(iii) $e = eP$

(iv) $0 < e_i < \infty \quad \forall i \in S$.

Remark: (1) Consider an irreducible, positive recurrent DTMC.

Then, $m_k < \infty$.

Set $\pi_i = \dfrac{e_i}{m_k} \quad \forall i$.

Then, $\pi = \pi P$ & π is a distribution

$\Rightarrow$ π is a stationary distribution.

(2) Consider an irreducible, positive recurrent DTMC.

$$\pi_i = \frac{e_i}{m_k}$$ ⟵ From part (1) above

$$\pi_k = \frac{1}{m_k}$$ ⟵ From main result (b)

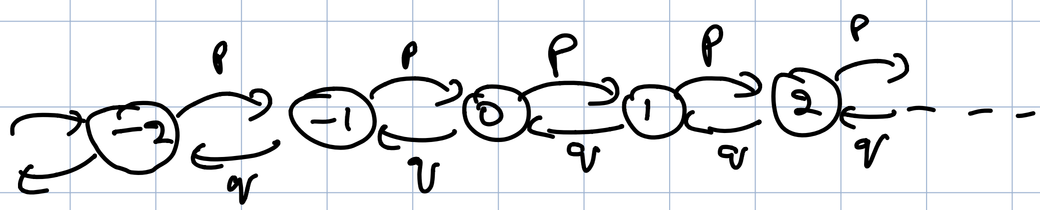
So, $e_i = \dfrac{\pi_i}{\pi_k}$

Given a stationary distribution (which is unique – see part b-(iii) of main result above), we have

$$\begin{array}{l}\text{expected \# visits to } i \\ \text{between visits to } k\end{array} = \frac{\pi_i}{\pi_k}$$

Lecture-21

Example:

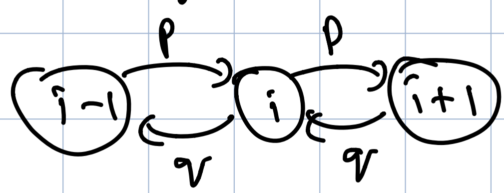


Irreducible $0 < p, q < 1$

A stationary measure: $e_i = 1 \ \forall i$

$$e_i = \sum_j e_j P_{j,i}$$

For any i,



$$q\, e_{i+1} + p\, e_{i-1} = e_i$$

Using $e_{i+1} = e_{i-1} = e_i = 1$, this is the same as $q + p = 1$, which holds.

So, $e_i = 1 \ \forall i$ is a stationary measure.

But, we cannot normalize this measure.

A bunch of claims: For an irreducible, recurrent DTMC, with ρ as constructed above, we have

(i) $\rho_k = 1$, (ii) $\sum_{i \in S} \rho_i = m_k$, (iii) $\rho = \rho P$

(iv) $0 < \rho_i < \infty \quad \forall i \in S$.

Proof: (i) Recall $W_i = \sum_{m=1}^{T_k} I(X_m = i)$ & $\rho_i = E(W_i \mid X_0 = k)$

So, $W_k = 1$ & $\rho_k = 1$.

(ii) $T_k = \sum_{i \in S} W_i$

$$m_k = E(T_k \mid X_0 = k)$$

$$= E\left(\sum_{i \in S} W_i \,\middle|\, X_0 = k\right)$$

monotone convergence theorem (See Sec 5.6 in Grimmett-Stirzaker's book, in particular, eq (13) there)

$$= \sum_{i \in S} E(W_i \mid X_0 = k)$$

$$= \sum_{i \in S} \rho_i$$

(iii) To show $\rho = \rho P$

$$\rho_j = E\left(\sum_{m=1}^{T_k} I(X_m = j) \,\middle|\, X_0 = k\right)$$

Let $E_k(\cdot) = E(\cdot \mid X_0 = k)$ & $P_k = P(\cdot \mid X_0 = k)$

So, $e_j = E_k\left(\sum_{m=1}^{T_k} I(X_m = j) \right)$

$$= E_k\left(\sum_{m=1}^{\infty} I(X_m = j, T_k \geq m) \right)$$

$$e_j = \sum_{m=1}^{\infty} P_k(X_m = j, T_k \geq m)$$

Notice that

$$P_k(X_m = j, T_k \geq m)$$

$$= \sum_{i \in S} P_k(X_{m-1} = i, X_m = j, T_k \geq m)$$

$$= \sum_{i \in S} P_k(X_m = j \mid X_{m-1} = i, T_k \geq m) P_k(X_{m-1} = i, T_k \geq m)$$

we strong Markov property (see detour) below

$$= \sum_{i \in S} P_{i,j} \, P_k(X_{m-1} = i, T_k \geq m)$$

holds by a stronger version of Markov property (cf. Sec 12.7 of Grimmett-Welsh's book)

A detour:

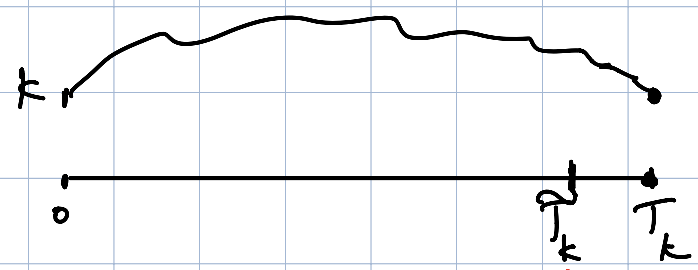
A r.v. $T : \Omega \to \{0, 1, 2, \dots\} \cup \{\infty\}$ is called a stopping time for a DTMC $\{X_n\}$ if $\forall n \geq 0$, the event $\{T = n\}$ is given in terms of $X_0, \dots X_n$.

Examples: First passage time is a stopping time.

$$\{T_k = n\} = \{X_0 = k, X_1 \neq k, \ldots, X_{n-1} \neq k, X_n = k\}$$

So, $\{T_k = n\}$ is determined by $X_0, \ldots, X_n$

Suppose $\tilde{T}_k = T_k - 1$



$\{\tilde{T}_k = n\}$ requires knowledge of X_{n+1}

one step before hitting state k

$$\tilde{T}_k = n \iff X_0 = k, X_1 \neq k, X_2 \neq k, \ldots, X_n \neq k, X_{n+1} = k$$

To know if $\{\tilde{T}_k = n\}$, we need knowledge of X_{n+1}

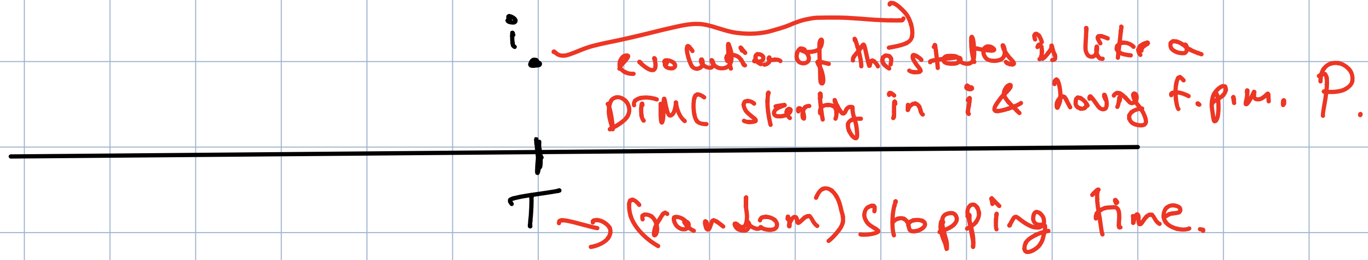
Hence, $\tilde{T}_k$ is NOT a stopping time.

Strong Markov property:

$\{X_n\}$ DTMC, t.p.m. P, T stopping time satisfying $T < \infty$

Given $X_T = i$, the sequence $(X_T, X_{T+1}, \ldots)$ is a Markov chain with initial state "i" & t.p.m. P.

Moreover, given $X_T = i$, $(X_T, X_{T+1}, \ldots)$ is independent of $(X_0, \ldots, X_{T-1})$



<End of detour>

Back to proof of $e = eP$

We have,

$$P_k(X_m = j, T_k \geq m) = \sum_{i \in S} P_{i,j} \; P_k(X_{m-1} = i, T_k \geq m)$$

Using this in the expression for e_j, we obtain

$$e_j = \sum_{m=1}^{\infty} \sum_{i \in S} P_{i,j} \; P_k(X_{m-1} = i, T_k \geq m)$$

$$= \sum_{i \in S} \sum_{m=1}^{\infty} P_{i,j} \; P_k(X_{m-1} = i, T_k \geq m)$$

$$= \sum_{i \in S} P_{i,j} \sum_{m=1}^{\infty} P_k(X_{m-1} = i, T_k \geq m)$$

$r = m-1$

$$= \sum_{i \in S} P_{i,j} \sum_{r=0}^{\infty} P_k(X_r = i, T_k \geq r+1)$$

$$= \sum_{i \in S} P_{i,j} \; E_k\left(\sum_{r=0}^{\infty} I\{X_r = i, T_k - 1 \geq r\}\right)$$

$$= \sum_{i \in S} P_{i,j} \; E_k\left(\sum_{r=0}^{T_k - 1} I\{X_r = i\}\right)$$

$$e_j = \sum_{i \in S} P_{i,j} \, e_i, \quad \text{i.e.,} \quad e = eP$$

Lecture-22

Proof of (iv) i.e.,

$$0 < e_i < \infty \quad \forall i.$$

Recall e was constructed using state k.

So, $e_k = 1$

Fix $i \in S$.

Irreducible $\Rightarrow$ $\exists\, m, n \geq 0$ s.t.

$$p^{(m)}_{i,k} > 0 \quad \& \quad p^{(n)}_{k,i} > 0$$

$e = eP$ (from part (iii))

$= eP^2$

$\vdots$

$$e = eP^m \quad \& \quad e = eP^n$$

$$e_k = \sum_j e_j\, p^{(m)}_{j,k} \qquad\qquad e_i = \sum_j e_j\, p^{(n)}_{j,i}$$

$$e_k \geq e_i\, p^{(m)}_{i,k} \qquad\qquad e_i \geq e_k\, p^{(n)}_{k,i}$$

Using $e_k = 1$, we have

$$e_i \leq \frac{1}{p^{(m)}_{i,k}} < \infty \qquad\qquad e_i \geq p^{(n)}_{k,i} > 0$$

$$\Rightarrow \quad 0 < e_i < \infty \quad \blacksquare$$

Big facts about stationary distributions

Irreducible DTMC. (not necessarily finite)

(i) If some state is positive recurrent, then $\exists$ a stationary distribution.

Pf: Suppose k is positive recurrent
$\Rightarrow \quad m_k < \infty$

From earlier thm, we have a stationary measure ρ i.e., $\rho = \rho P$

constructed using state k

Set $\pi_i = \dfrac{\rho_i}{m_k}$, then

$\pi = \pi P$ & also π is a distribution

Since, $\sum \pi_i = \dfrac{\sum \rho_i}{m_k} = 1$

(ii) If $\exists$ a stationary distribution π, then

(a) each state is positive recurrent

(b) $\pi_i = \dfrac{1}{m_i} \quad \forall i$ (c) π is unique (follows from (b))

Pf:

Claim: $\pi_i > 0, \ \forall i \in S$

Pf inside Pf:

$$\sum_i \pi_i = 1$$

So, $\exists$ at least one state, say k, with $\pi_k > 0$.

Fix $i \in S$.

Irreducibility $\Rightarrow$ $\exists m$ s.t. $p_{k,i}^{(m)} > 0$

$$\pi = \pi P = \pi P^2 = \dots\dots = \pi P^m$$

$$\pi_i = \sum_j \pi_j \, p_{j,i}^{(m)} \geq \pi_k \, p_{k,i}^{(m)} > 0$$

$\pi_i > 0$

(End of Pf within Pf)

Irreducible $\Rightarrow$ all states are either transient or recurrent.

In an earlier lecture, it was shown that for a transient irreducible DTMC,

$$p_{i,j}^{(n)} \to 0 \text{ as } n \to \infty \quad \forall i, j \quad \text{—— } (*)$$

Using $(*)$ & $\pi_i > 0 \ \forall i$ & $\pi = \pi P$, we shall arrive at a contradiction.

"Assume all states are transient."

$$\pi_j = \sum_i \pi_i P_{i,j}^{(n)} \xrightarrow[n\to\infty]{} 0$$

(for the finite case, interchange of lim & Σ is easy to see. General case, use the argument on p. 30 of this note).

A contradiction since $\pi_j > 0$

What we got: If $\exists$ a stationary distribution π in an irreducible DTMC, then all states are recurrent.

What is left to prove: Establishing positive recurrence & $\pi_i = 1/m_i$.

<u>Pf of whatever is left to be proven:</u>

Let $X_0 \sim \pi$ ← stationary

Then,

$$(X_1, \ldots, X_{n-1}) \sim (X_0, \ldots, X_{n-2})$$

have the same distribution.

Now, $\pi_i m_i$

$$= P(X_0 = i) \sum_{n=1}^{\infty} P(T_i \geq n \mid X_0 = i)$$

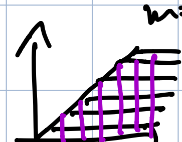
$$= \sum_{n=1}^{\infty} P(T_i \geq n, X_0 = i)$$

$X \geq 1$ discrete r.v.

$$EX = \sum_{n=1}^{\infty} P(X \geq n)$$

$$EX = \sum_m m\, p_m$$

$$= \sum_{m=0}^{\infty} \left(\sum_{l=0}^{m} 1\right) p_m$$



$$= \sum_{l=0}^{\infty} \sum_{m=l}^{\infty} p_m$$

$$P(T_i \geq 1, X_0 = i) = P(X_0 = i) \text{ — } (*)$$

$= \sum_{l=0}^{\infty} P(X \geq l)$

since $T_i = \min\{k \geq 1 \mid X_k = i\}$

For $n \geq 2$

$X_0 = i$ ... X_{n-1} X_n

i is not visited since $T_i \geq n$

$$P(T_i \geq n, X_0 = i)$$
$$= P(X_0 = i, X_1 \neq i, X_2 \neq i, \dots, X_{n-1} \neq i)$$
$$= P(X_0 = i, X_m \neq i, 1 \leq m \leq n-1)$$
$$= P(X_m \neq i, 1 \leq m \leq n-1) - P(X_m \neq i, 0 \leq m \leq n-1)$$

Using $(X_1, \dots, X_{n-1}) \sim (X_0, \dots, X_{n-2})$

$$= P(X_m \neq i, 0 \leq m \leq n-2) - P(X_m \neq i, 0 \leq m \leq n-1)$$

Setting $\alpha_r = P(X_m \neq i, 0 \leq m \leq r)$

$$= \alpha_{n-2} - \alpha_{n-1} \text{ —— } (**)$$

Going back to $\pi_i m_i$, using $(*)$ & $(**)$, we obtain

$$\pi_i m_i = \sum_{n=1}^{\infty} P(T_i \geq n, X_0 = i)$$
$$= P(X_0 = i) + \sum_{n=2}^{\infty} (\alpha_{n-2} - \alpha_{n-1})$$
$$= P(X_0 = i) + \alpha_0 - \lim_{n \to \infty} \alpha_n$$
$$= P(X_0 = i) + P(X_0 \neq i) - \lim_{n \to \infty} \alpha_n$$
$$\pi_i m_i = 1 - \lim_{n \to \infty} \alpha_n$$

$$\lim_{n\to\infty} \alpha_n = P(X_m \neq i, \forall m)$$

$$= 0 \qquad \text{since } i \text{ is recurrent}$$

So, $\pi_i m_i = 1$

i.e., $m_i = \dfrac{1}{\pi_i} < \infty$

So, i is positive recurrent.

Irreducibility $\Rightarrow$ all states are positive recurrent.

Uniqueness of π follows from $\pi_i = \dfrac{1}{m_i} \; \forall i$

$\blacksquare$

Lecture - 23

Convergence to stationary distribution.

Big Fact I:

Let $\{X_n, n \geq 0\}$ be an irreducible, aperiodic & positive recurrent DTMC. Then,

$$P_{i,j}^{(n)} \longrightarrow \pi_j \quad \text{as } n \to \infty \quad \forall i, j,$$

where π is the "unique" stationary distribution.

Pf: Uses "Coupling".

Let $\{Y_n\}$ be an independent copy of the given DTMC, i.e., $\{Y_n\}$ has the same state, say S, space & t.p.m. P as $\{X_n\}$.

$$Z_n = (X_n, Y_n)$$

Is Z_n a DTMC? Yes.

State space $S \times S$

t.p.m. for Z_n:

$$P_{ij,kl} = P\left(Z_{n+1} = (k,l) \mid Z_n = (i,j) \right)$$

$\{X_n\}$ & $\{Y_n\}$ independent

$$= P\left(X_{n+1} = k \mid X_n = i\right) P\left(Y_{n+1} = l \mid Y_n = j\right)$$

$$= P_{i,k} \; P_{j,l}$$

Is Z_n irreducible? Yes.

$\{X_n\}, \{Y_n\} \leftarrow$ irreducible, aperiodic (Given)

$\exists N$ s.t. $\forall n \geq N$

$$P^{(n)}_{i,k} \; P^{(n)}_{j,l} > 0 \quad \forall n \geq N$$

H.w. proof requires aperiodicity.

Does $\{Z_n\}$ have a stationary distribution? Yes.

Let $\nu = (\nu_{i,j},\ i,j \in S \times S)$ denote the stationary distribution of $\{Z_n\}$.

Thus, $\quad \nu_{i,j} = \pi_i \pi_j \quad$, where

π is the stationary distribution of $\{X_n\}$

So, $\{Z_n\}$ is positive recurrent.

Coupling trick:

Suppose $X_0 = i$ & $Y_0 = j$

$Z_0 = (i,j)$

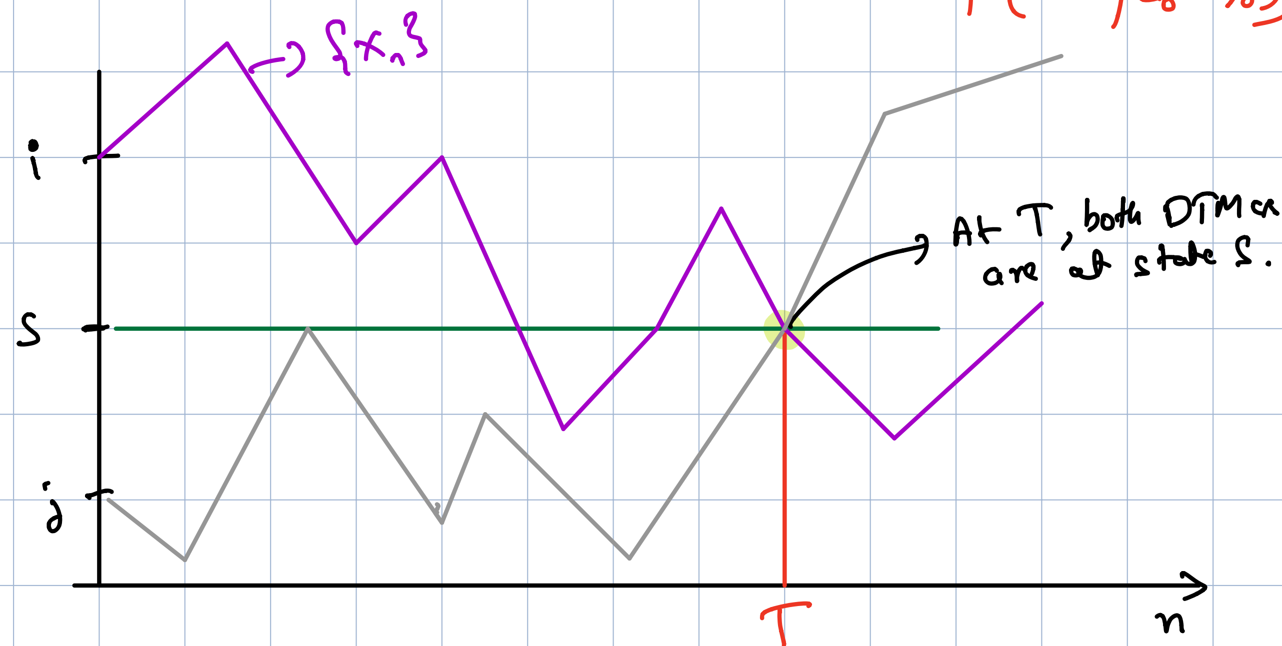
Fix some state $s \in S$.

$T = \min\{n \geq 1 \mid Z_n = (s,s)\}$

first hitting time of (s,s) in $\{Z_n\}$

$$\underbrace{P(T < \infty \mid Z_0 = (i,j))}_{\text{denote this by } P_{ij}(T<\infty)} = 1 \quad \text{—— } (*)$$

denote this by $P_{ij}(T<\infty)$, i.e., $P_{ij}(\cdot) = P(\cdot \mid Z_0 = (i,j))$



Is T a stopping time? Yes.

Conditional on $\{T \le n\}$, both X_n & Y_n have the same distributions. ← Strong Markov property

If two indep. DTMCs with identical t.p.m. start in the same state, say $X_0 = s$, $Y_0 = s$, then the distribution of X_n & Y_n are going to be the same. A similar observation holds for a stopping time T, when one looks at the event $\{T \le n\}$

Strong Markov property →

$$P^{(n)}_{i,k} = P_{ij}(X_n = k)$$

$$= P_{ij}(X_n = k, T \le n) + P_{ij}(X_n = k, T > n)$$

with $\{T \le n\}$, X_n & Y_n have the same distribution

$$= P_{ij}(Y_n = k, T \le n) + P_{ij}(X_n = k, T > n)$$

$$\le P_{ij}(Y_n = k) + P_{ij}(T > n)$$

$$= P^{(n)}_{j,k} + P_{ij}(T > n)$$

So, $$P^{(n)}_{i,k} - P^{(n)}_{j,k} \le P_{ij}(T > n)$$

Swap i & j & repeat the argument above to get

$$P^{(n)}_{j,k} - P^{(n)}_{i,k} \le P_{ij}(T > n)$$

Or, $$\left| P^{(n)}_{i,k} - P^{(n)}_{j,k} \right| \le P_{ij}(T > n)$$

As $n \to \infty$, $P_{ij}(T > n) \to 0$ because of $(*)$

So, $p_{i,k}^{(n)} - p_{j,k}^{(n)} \to 0$ as $n \to \infty$.

So, if $\lim_{n\to\infty} p_{i,k}^{(n)}$ exists, then it does not depend on starting state i.

$\lim_{n\to\infty} p_{i,k}^{(n)}$ exists because

$$\pi_k - p_{j,k}^{(n)}$$

$$= \sum_i \pi_i \left(p_{i,k}^{(n)} - p_{j,k}^{(n)} \right) \quad \text{since } \pi = \pi P \ \& \ \sum \pi_i = 1$$

$$\xrightarrow{n\to\infty} 0$$ (by a version of dominated convergence theorem. See p.30 for instance).

So, $p_{j,k}^{(n)} \to \pi_k$ as $n \to \infty$.

$\blacksquare$

Lecture-24

Remark: A version of "Big fact 2" holds for the case of null recurrent + irreducible + aperiodic DTMC, i.e.,

$$p_{i,j}^{(n)} \to 0 \quad \text{as } n\to\infty \qquad \text{—— } (*)$$

For a proof, see Grimmett & Stirzaker, Sec 6.4.

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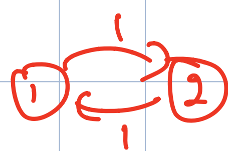
* For positive/null recurrent case, one needs aperiodicity to infer $(*)$.

An easy counterexample:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



"Convergence in Cesaro sense".

$$\tilde{V}_n(j) = \sum_{m=1}^{n} I(X_m = j)$$ ← occupancy measure for state j

Time average $= \frac{1}{n}\tilde{V}_n(j)$

As $n\to\infty$, where does the "time average" converge, for a irreducible, positive recurrent DTMC?

$$\frac{1}{n}\tilde{V}_n(j) \longrightarrow \pi_j \quad \text{as } n\to\infty \quad \text{a.s. (w.p.1)}$$

or equivalently, $\frac{1}{n}\tilde{V}_n(j) \to \frac{1}{m_j}$ as $n \to \infty$ w.p. 1.

For a null recurrent state j,

$$\frac{1}{n}\tilde{V}_n(j) \longrightarrow 0 \quad \text{a.s.} \quad \text{as } n \to \infty.$$

* These claims do not require aperiodicity.

Big Fact II: Consider a irreducible, recurrent DTMC. Then, for any $j \in S$,

$$\frac{\tilde{V}_n(j)}{n} \longrightarrow \frac{I\{T_j < \infty\}}{m_j} \quad \text{as } n \to \infty \quad \text{w.p. 1}$$

(almost surely = a.s.) $\Updownarrow$

for any start state $X_0 = i$.

A few remarks:

① $$E\left(\frac{\tilde{V}_n(j)}{n} \middle| X_0 = i\right) = \frac{1}{n}\sum_{m=1}^{n} E\left(I(X_m = j) \middle| X_0 = i\right)$$

$$= \frac{1}{n}\sum_{m=1}^{n} P^{(m)}(i,j)$$

$$E\left(\frac{I\{T_j < \infty\}}{m_j} \middle| X_0 = i\right) = \frac{P(T_j < \infty \mid X_0 = i)}{m_j}$$

So, using DCT,

$$\frac{1}{n}\sum_{m=1}^{n} P^{(m)}(i,j) \xrightarrow{n \to \infty} \frac{P(T_j < \infty \mid X_0 = i)}{m_j} \quad \text{— } (**)$$

In (**), if j is null recurrent, then the limit is zero.

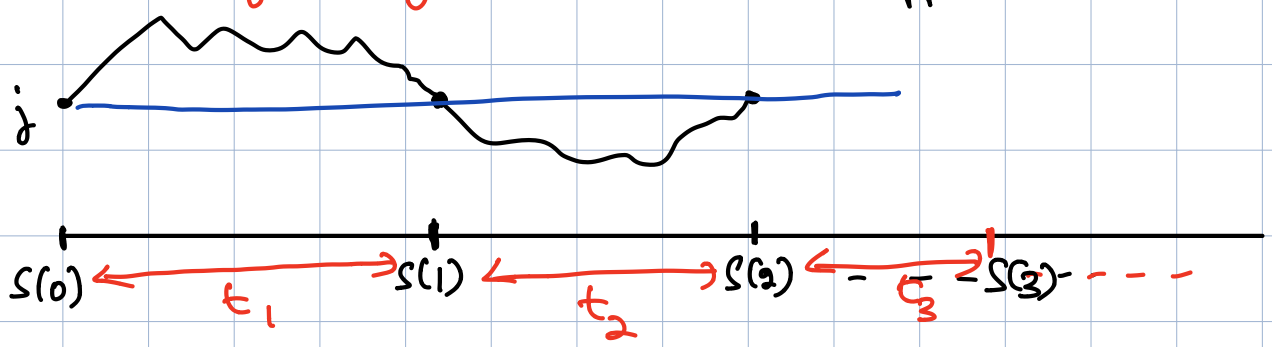
In (**), if j is positive recurrent, then the limit is positive.

② For a transient state j, $\sum_{m=1}^{\infty} p^m(i,j) < \infty$.

$$\frac{1}{n} \sum_{m=1}^{\infty} p^m(i,j) \xrightarrow{n\to\infty} 0$$

i.e., $$E\left(\frac{\tilde{V}_n(j)}{n} \,\middle|\, X_0 = i \right) \xrightarrow{n\to\infty} 0$$ for transient j.

Proof of Big Fact II: Suppose $X_0 = j$.



$S(k) = \min\{n \geq 1 \mid \tilde{V}_n(j) = k\}$, $S(0) = 0$.

$t_k = S(k) - S(k-1)$

Sequence $t_1, t_2, t_3, \ldots\ldots$ is an iid sequence.

$$E(t_1 \mid X_0 = j) = m_j$$

SLLN: $X_1, X_2, \ldots$ iid non-negative r.v.s with mean $\mu < \infty$. $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, $\bar{X}_n \to \mu$ as $n \to \infty$ w.p. 1.

$$\frac{1}{k}\sum_{i=1}^{k} t_i \xrightarrow[k\to\infty]{} m_j \quad \text{a.s. under } P(\cdot \mid X_0 = j)$$

or,

$$\frac{S(k)}{k} \to m_j \qquad \text{———} \qquad (***)$$

$$S(\tilde{V}_n(j)) \le n < S(\tilde{V}_n(j)+1)$$

$$\frac{S(\tilde{V}_n(j))}{\tilde{V}_n(j)} \le \frac{n}{\tilde{V}_n(j)} < \frac{S(\tilde{V}_n(j)+1)}{\tilde{V}_n(j)+1} \times \frac{(\tilde{V}_n(j)+1)}{\tilde{V}_n(j)}$$

$$\underbrace{\frac{S(\tilde{V}_n(j))}{\tilde{V}_n(j)}}_{\downarrow\, m_j} \le \frac{n}{\tilde{V}_n(j)} < \underbrace{\frac{S(\tilde{V}_n(j)+1)}{\tilde{V}_n(j)+1}}_{\downarrow\, m_j} \times \underbrace{\left(1 + \frac{1}{\tilde{V}_n(j)}\right)}_{\downarrow\, 1}$$

As $n\to\infty$, 1 since j is recurrent $\tilde{V}_n(j) \to \infty$ as $n \to \infty$.

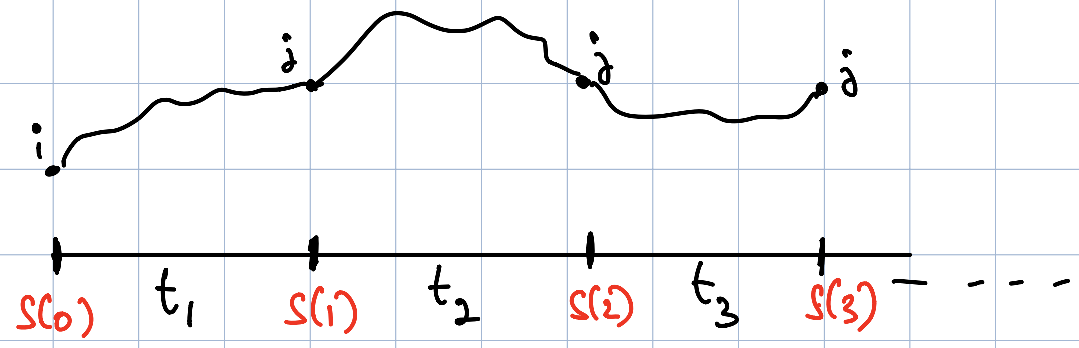
So, $\frac{n}{\tilde{V}_n(j)} \to m_j$ a.s. as $n \to \infty$ when $X_0 = j$.

Next, we consider the case $X_0 = i$ & $i \ne j$.

If $\{T_j = \infty\}$, then $\tilde{V}_n(j) = 0$

$$\frac{\tilde{V}_n(j)}{n} \to 0 \quad \text{on } \{T_j = \infty\}$$

If $\{T_j < \infty\}$



$t_1, t_2, t_3, \ldots$ are independent.

But $t_2, t_3, \ldots\ldots$ are iid.

$$\frac{S(k)}{k} = \frac{t_1}{k} + \frac{t_2 + \cdots + t_k}{k}$$

$$\xrightarrow[k\to\infty]{} 0 + m_j \quad \text{a.s.}$$

So, $\frac{S(k)}{k} \longrightarrow m_j$ a.s. under $P(\cdot \mid X_0 = i)$

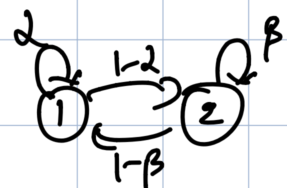
This is the equivalent of $(***)$ for $X_0 = i$, $i \neq j$.
Follow the steps below $(***)$, i.e., the sandwich argument, to infer

$$\frac{n}{V_n(j)} \xrightarrow{n\to\infty} m_j \quad \text{a.s. under } P(\cdot \mid X_0 = i)$$

The claim follows. $\blacksquare$

Some examples:

① Two state DTMC:



$\alpha + \beta < 2$

$$\pi = \pi P \iff \begin{aligned} \pi_1 &= \alpha \pi_1 + (1-\beta)\pi_2 \\ \pi_2 &= (1-\alpha)\pi_1 + \beta \pi_2 \end{aligned}$$

$$\pi_1 + \pi_2 = 1$$

If you solve, then

$$\pi_1 = \frac{1-\beta}{2-\alpha-\beta}, \quad \pi_2 = \frac{1-\alpha}{2-\alpha-\beta}.$$

Check the finding in Example 1 on p. 2 of this note.

H.W. Find stationary π for the periodic DTMC in Example 2 on p. 3. of this note.

Lecture - 25

A few examples.

① Success runs

$$P_{i,i+1} = p_i \,, \quad P_{i,0} = q_i \,, \quad p_i, q_i > 0 \,, \forall i$$

Irreducible & aperiodic (Check this)

$$(i) \xrightarrow{p_i} (i+1)$$

$\pi = \pi P$:

$$\pi_{i+1} = p_i \, \pi_i$$

Unrolling, we obtain

$$\pi_i = p_0 \, p_1 \cdots p_{i-1} \, \pi_0$$

$$\pi_i = \rho_i \, \pi_0 \,, \quad \text{where} \quad \rho_i = \begin{cases} 1 & \text{if } i = 0 \\ p_0 p_1 \cdots p_{i-1} & \text{if } i \geq 1 \end{cases}$$

$$\sum_i \pi_i = 1$$

$$\pi_0 \left(\sum_{i=0}^{\infty} \rho_i \right) = 1$$

If $\sum \rho_i$ converges, then $\pi_0 = \dfrac{1}{\sum_{i=0}^{\infty} \rho_i}$

In this case, the DTMC is positive recurrent.

H.W.: If $\sum \ell_i = \infty$, determine null recurrence/transience of "success runs" DTMC.

Special case: $p_i = p \quad \forall i$

$$\ell_i = p^i \Rightarrow \sum \ell_i = \frac{1}{1-p} \Rightarrow \pi_0 = 1-p$$

$$\pi_i = p^i(1-p) \leftarrow \text{Geometric distribution.}$$

---

(2) Simplified version of Google Pagerank

Suppose there are 4 webpages

$$C = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$C(i,j) = 1$ if there is a link from page i to j

e.g. from page 1, we could end up in page 2 or page 4.

& page 4 got no links.

User behaviour: Visit these webpages in a Markovian fashion, i.e., pick one of the links unif. at random from a webpage & if there ain't any link, then pick any of the 4 webpages uniformly.

The t.p.m. of DTMC is

$$P = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.33 & 0 & 0.33 & 0.33 \\ 1 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

By construction, this DTMC is irreducible & has a finite state space $\Rightarrow$ positive recurrent $\Rightarrow \exists$ a stationary distribution.

For this toy example,

$$\pi = \begin{bmatrix} 0.308 & 0.231 & 0.152 & 0.308 \end{bmatrix}$$

Rank pages according to π, i.e., put page 1 & 4 at the top, followed by page 2 & page 3.

DTMCs with costs

Discounted Cost

Average Cost

DTMCs with discounted costs:

In addition to Markovian transitions, each state is associated with a cost.

State space S, For each $x \in S$, associate a cost $c(x)$. ← say bounded

Add a discount factor $0 < \alpha < 1$.

Objective $$C = \sum_{n=0}^{\infty} \alpha^n c(X_n)$$

↓ infinite horizon discounted cost.

For each state i, define $J(i) = E(C \mid X_0 = i)$

DTMCs with average cost:

With notation as before,

Objective $$C = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} E(c(X_m))$$

long-run avg. cost

$$\frac{1}{n} \sum_{m=1}^{n} E(c(X_m)) \rightarrow$$ average cost over a horizon of length n.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} E(c(X_m)) \rightarrow$$ long-run average cost.

Discounted vs. average cost:

Intuitively, discounted cost is concerned with transient behaviour, while average cost is concerned with limiting behaviour.

For e.g. at $n_0 = 100$, $\alpha = 0.9$, the contribution of the cost $c(X_n)$ $\forall n \geq n_0$ is negligible owing to the discount factor $(0.9)^{100}$.

In contrast,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} E(c(X_m)) = \lim_{n\to\infty} \left\{ \cancel{\frac{1}{n} \sum_{m=1}^{100} E(c(X_m))} \to 0 \right.$$

$$\left. + \frac{1}{n} \sum_{m\geq 100} E(c(X_m)) \right\}$$

So, avg. cost is concerned with the cost in the steady-state/ in the limit.

Lecture - 26

Question: How to compute discounted cost objective?

Recall $J(i) = E\left(\sum_{n=0}^{\infty} \alpha^n c(X_n) \,\middle|\, X_0 = i \right)$

Vector $c = [c(i)]_{i\in S}$

and vector $J = [J(i)]_{i\in S}$

Claim:

$$\underbrace{J}_{\text{discounted cost vector}} = (I - \underbrace{\alpha}_{\substack{\text{discount factor} \\ 0\leq\alpha<1}} \underbrace{P}_{\text{t.p.m.}})^{-1} \underbrace{c}_{\text{single-stage cost vector}}$$

Pf:

$$J(i) = E\left(\sum_{n=0}^{\infty} \alpha^n c(X_n) \,\Big|\, X_0 = i \right)$$

$$= E\Big[c(X_0) + \underbrace{\sum_{n=1}^{\infty} \alpha^n c(X_n)}_{\text{denote this as } C_1} \,\Big|\, X_0 = i \Big)$$

$$E(C_1 \mid X_1 = j) = E\left(\sum_{n=1}^{\infty} \alpha^n c(X_n) \,\Big|\, X_1 = j \right)$$

$$= E\left(\alpha \sum_{m=0}^{\infty} \alpha^m c(X_m) \,\Big|\, X_0 = j \right)$$

$$= \alpha J(j)$$

$$J(i) = c(i) + \alpha E(J(j) \mid X_0 = i)$$

$$J(i) = c(i) + \alpha \sum_{j \in S} P_{i,j} J(j)$$

In vector-matrix notation,

$$J = c + \alpha P J$$

$$\text{(or)} \quad J = (I - \alpha P)^{-1} c.$$

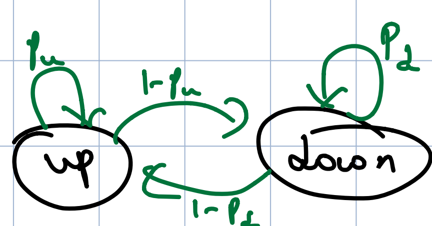
$x = f(x)$ fixed point eqn

$J = TJ$, where $TJ = c + \alpha P J$

$\blacksquare$

Example:

Machine → States: up, down



$$P = \begin{bmatrix} p_d & 1-p_d \\ 1-p_u & p_u \end{bmatrix}$$

Costs: $c = \begin{bmatrix} c(\text{down}) \\ c(\text{up}) \end{bmatrix} = \begin{bmatrix} d \\ -r \end{bmatrix}$

Discounted cost

$$J = \begin{bmatrix} J(\text{down}) \\ J(\text{up}) \end{bmatrix} = (\underline{I} - \alpha P)^{-1} c$$

$$J = \frac{1}{\text{const}} \begin{bmatrix} d(1-\alpha p_u) - r\alpha(1-p_d) \\ d\alpha(1-p_u) - r(1-\alpha p_d) \end{bmatrix}, \text{ where}$$

$$\text{const} = (1-\alpha p_u)(1-\alpha p_d) - \alpha^2(1-p_u)(1-p_d).$$

"$-J(\text{up})$" $\rightarrow$ expected infinite horizon revenue for a machine that is "up".

$$-J(\text{up}) = \frac{-d\alpha(1-p_u) + r(1-\alpha p_d)}{\text{const.}}$$

Suppose a new machine costs "m".
Then, it is worth buying a new machine if

$$m \leq -J(\text{up}).$$

Spl. case: $\alpha = 0$: condition above $\Leftrightarrow$ $m \leq r$.

Question: How to compute long run avg. cost objective?

long. run avg cost $\quad J(i) = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} E(c(X_m) \mid X_0 = i)$

(*) Suppose DTMC is irreducible, positive recurrent
($\Rightarrow$ $\exists$ a stationary distribution π)

What we will show is $J(i) \equiv J \quad \forall i.$

<u>Claim:</u> For a DTMC satisfying (*), with
$\sum_j \pi_j |c(j)| < \infty$, we have

$$J(i) = \sum_{j\in S} \pi_j \, c(j)$$

$\leftarrow$ avg. cost same for all start states.

<u>Pf:</u>

$$J(i) = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} E(c(X_m) \mid X_0 = i)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j\in S} V_{i,j}^{(n)} c(j),$$

where $V_{i,j}^{(n)}$ = expected # visits to j in $\{1 \ldots n\}$ starting in state i

So, $$J(i) = \lim_{n\to\infty} \sum_{j\in S} \frac{V_{i,j}^{(n)}}{n} c(j)$$

$$= \sum_{j\in S} \left(\lim_{n\to\infty} \frac{V_{i,j}^{(n)}}{n} \right) c(j)$$

$\leftarrow$ use dominated convergence theorem to justify lim & Σ interchange.

$$J(i) = \sum_{j \in S} \pi_j \, c(j). \quad \blacksquare$$

← we used positive recurrence + Cesaro convergence (See big fact II)

For average cost objective, a good reference is "Dynamic programming & optimal Control, Vol. II", by D.P. Bertsekas, Athena Scientific.

__Example:__ Brand switching

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \end{array}$$

e.g. $P(\text{buying } B \mid \text{given bought } A \text{ last time}) = 0.2$

Costs: $c(A) = 6$, $c(B) = 5$, $c(C) = 4$.

Finite irreducible DTMC $\Rightarrow$ $\exists \pi$ s.t. $\pi = \pi P$, $\sum \pi_i = 1$

$$\pi_A = 0.13, \quad \pi_B = 0.32, \quad \pi_C = 0.55$$

Long run average cost $J = \pi_A c(A) + \pi_B c(B) + \pi_C c(C)$
$= 4.58$

So, the customer spends 4.58 <whatever currency> on each visit in the long run.

<End of DTMCs>
