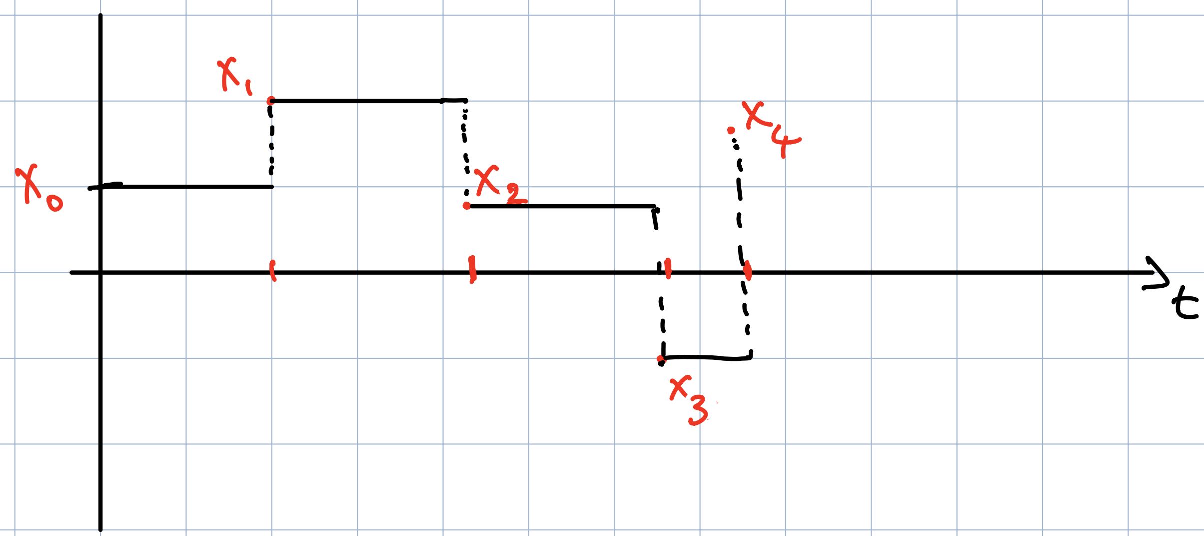
Lecture-27

Continuous time Markov chains

Ref: Kulkarni's book, chapter 6.

DTMC: state transitions occured at discrete points $\{X_0, X_1, X_2 \dots\}$

CTMC: continuous time process $\{X(t), t \geq 0\}$



Def: A process $\{X(t), t \geq 0\}$ with a countable state space S, is a CTMC if

$$P(X(t+s)=j \mid X(s)=i, X(u): 0 \leq u < s)$$
$$= P(X(t+s)=j \mid X(s)=i)$$ Markov property

$$P(X(t+s)=j \mid X(s)=i) = P(X(t)=j \mid X(0)=i)$$ Time homogeneity.

We shall study "Time-homogeneous CTMCs".

The CTMC evolution!

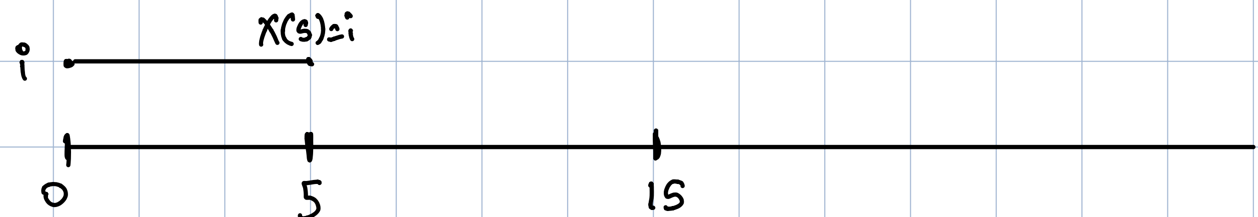
Suppose the CTMC is in state i at some time instant, say 0.

Suppose CTMC remains in "i" during $[0,5]$

What is the probability that it will remain in i during $[5,15]$ given that it was in "i" during $[0,5]$?

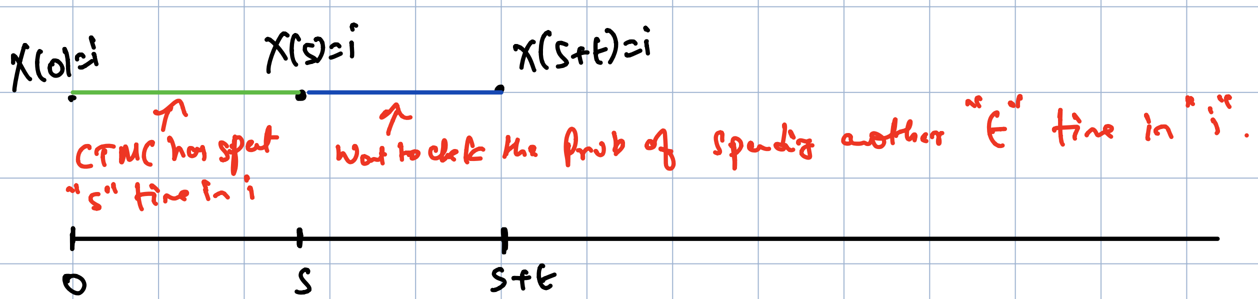
Let $T_i \to$ denote time spent in i

$$P(T_i > 15 \mid T_i > 5) = P(T_i > 10)$$



Suppose CTMC is in state i at time s.

$$T_i = \min\{t \geq 0 \mid X(s+t) \neq i\}$$



If $\quad P(T_i > s+t \mid T_i > s) = P(T_i > t)$

then $\quad T_i \sim$ exponentially distributed.

$$P(T_i > s+t \mid T_i > s)$$

$$= P(X(u)=i,\ u \in [0,s+t] \mid X(u)=i,\ u \in [0,s])$$

$$= P(X(u)=i,\ u \in [s,s+t] \mid X(u)=i,\ u \in [0,s])$$

Markov property

$$= P(X(u)=i,\ u \in [s,s+t] \mid X(s)=i)$$

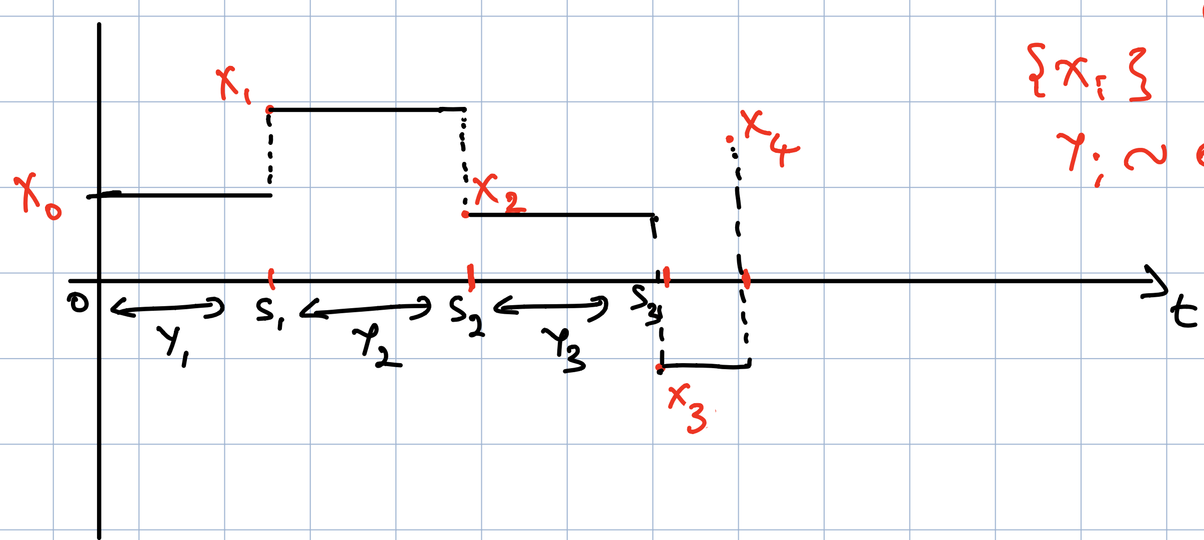
Time homogeneity

$$= P(X(u)=i,\ u \in [0,t] \mid X(0)=i)$$

$$= P(T_i > t)$$

So, T_i is exponentially distributed.

Alternate characterization of CTMC:



(Embedded)
$\{X_i\}$ DTMC

$Y_i \sim$ exponential r.v.s

Spl. case: $X_i = X_{i-1} + 1$ & $Y_i \sim Exp(\lambda)$ leads to Poisson process with rate λ.

For a CTMC $\{X(t), t \geq 0\}$, we have

$$P(X_{n+1} = j, Y_{n+1} > y \mid X_n = i, Y_n, X_{n-1}, \dots\dots, X_1, Y_1, X_0)$$

$$= P(X_1 = j, Y_1 > y \mid X_0 = i)$$

$$= P_{i,j} \exp(-q_i y). \qquad \forall i,j \in S,\ y \geq 0,\ \forall n$$

where $\sum_{j \neq i} P_{i,j} = 1$, $P_{i,j} \geq 0$. [Note $P_{i,i} = 0$], $\forall i$.

Example:

1) Two-state machine

$$\text{machine} \begin{cases} \to \text{up} \\ \to \text{down} \end{cases}$$

When up, machine spends $\exp(\mu)$ time before going down.
& when down, $\exp(\lambda)$ time to get repaired & come back up.

Let 0 denote down & 1 denote up.
CTMC specification:

$$X(t) \in \{0, 1\}$$

$$P_{0,1} = 1, \quad P_{1,0} = 1$$

$$q_0 = \lambda, \quad q_1 = \mu$$

2) $PP(\lambda)$ is a CTMC

$$S = \{0, 1, 2, \ldots\}$$

$$P_{i,i+1} = 1 \ , \quad v_i = \lambda \ , \quad \forall i$$

What is enough to characterize a CTMC?

$\{X(t), t \geq 0\}$ CTMC

$$P_{i,j}(t) = P(X(t) = j \mid X(0) = i) \ , \quad \forall i, j, t \geq 0$$

$$\underbrace{P(t) = [[P_{i,j}(t)]]_{i,j \in S},}_{\text{Need this t.p.m. for every } t.} \quad t \geq 0$$

Initial distribution "a": $\quad a_i = P(X(0) = i)$

$$a = [[a_i]]_{i \in S}.$$

Using "a" & $P(t)$, we can find the finite dimensional distributions of the CTMC as follows:

Fix $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, $i_1, \ldots i_n \in S$

$$P(X(t_1)=i_1, \dots\dots\dots, X(t_n)=i_n)$$

$$= \sum_{i_0 \in S} P(X(t_1)=i_1, \dots\dots\dots, X(t_n)=i_n \mid X(0)=i_0)\, a_{i_0}$$

$$= \sum_{i_0 \in S} a_{i_0}\, P(X(t_2)=i_2, \dots, X(t_n)=i_n \mid X(0)=i_0, X(t_1)=i_1)\, P_{i_0,i_1}(t_1)$$

$$= \sum_{i_0 \in S} a_{i_0}\, P_{i_0,i_1}(t_1)\, P(X(t_2)=i_2, \dots, X(t_n)=i_n \mid X(t_1)=i_1)$$

$$= \sum_{i_0 \in S} a_{i_0}\, P_{i_0,i_1}(t_1)\, P(X(t_2-t_1)=i_2, \dots, X(t_n-t_1)=i_n \mid X(0)=i_1)$$

$$\vdots$$

$$= \sum_{i_0 \in S} a_{i_0}\, P_{i_0,i_1}(t_1)\, P_{i_1,i_2}(t_2-t_1) \dots\dots\dots P_{i_{n-1},i_n}(t_n-t_{n-1})$$

The finite dimensional distributions of a CTMC are given by

$$P(X(t_1)=i_1, \dots\dots\dots, X(t_n)=i_n) = \sum_{i_0 \in S} a_{i_0}\, P_{i_0,i_1}(t_1)\, P_{i_1,i_2}(t_2-t_1) \dots\dots\dots P_{i_{n-1},i_n}(t_n-t_{n-1})$$

From the claim above, one requires (i) initial distribution

& (ii) $\underbrace{P(t), \ \forall t \geq 0}_{\text{hard part.}}$ to specify the CTMC.

So, we use the alternate characterization i.e., embedded DTMC + exponential rates to specify a CTMC.

CTMC model:

$E_{i,j}$ $(i \neq j)$: occurs if the CTMC transitions from i to j.

Suppose $X(0) = i$

i ——— i' $\qquad i' \in S \setminus \{i\}$

$\leftarrow \exp(q_i)$ time

i ——— j_1

$T_{i,j_1} \sim \exp(q_{i,j_1})$

i ——— j_2

$T_{i,j_2} \sim \exp(q_{i,j_2})$

$\vdots$

i ——— $j_{|S|-1}$

$T_{i,j_{|S|-1}} \sim \exp(q_{i,j_{|S|-1}})$

Imagine there are $|S|-1$ counters (exponential) & the next state will be decided by the min of these counters.

$$T_i = \min\{T_{i,j_k},\ k = 1, \dots, |S|-1,\ j_k \neq i \ \forall k\}$$

Distribution of $T_i = \exp\left(\sum_{j \neq i} q_{i,j}\right)$

i.e., $T_i \sim \mathcal{E}xp(q_i)$ with $q_i = \sum_{j \neq i} q_{i,j}$

$$P(X_{n+1} = j, Y_{n+1} > y \mid X_n = i, Y_n, X_{n-1}, Y_{n-1}, \dots X_1, Y_1, X_0)$$

$$= P(T_{i,j} = T_i, \; T_i > y)$$

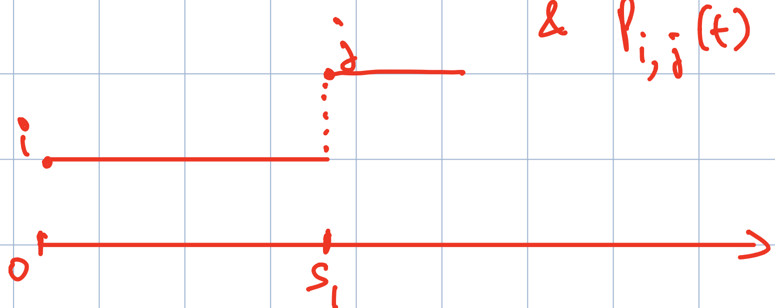
$$= \frac{q_{i,j}}{q_i} \exp(-q_i y)$$

Can be seen as the prob. of transitioning from i to j in the embedded DTMC.

$$= P_{i,j} \exp(-q_i y), \text{ where } P_{i,j} = \frac{q_{i,j}}{q_i},$$

$$\left(\text{Recall } q_i = \sum_{j \neq i} q_{i,j}\right)$$

[$P_{i,j}$ → t.p.m of DTMC. $P_{i,j}(t) = P(X(t) = j \mid X(0) = i)$
& $P_{i,j}(t) \neq P_{i,j}$ in general]



$$P(X(S_1^+) = j \mid X(0) = i) = P_{i,j}$$

Generator matrix Q:

Set $q_{i,i} = -\sum_{k \neq i} q_{i,k} = -q_i, \ \forall i$

$$Q = [[\, q_{i,j} \,]]_{i,j \in S}$$

row-sums of Q: zero.

Example: ① Two-state machine

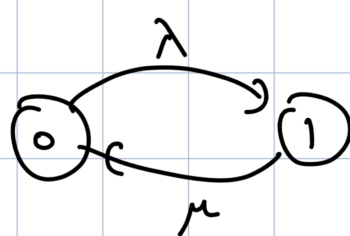
machine → up, → down | When up, machine spends $\exp(\mu)$ time before going down. & when down, $\exp(\lambda)$ time to get repaired & come back up.

$0 = $ down, $1 = $ up.

$T_{0,1} \sim \mathcal{E}xp(\lambda), \quad q_{0,1} = \lambda$

$T_{1,0} \sim \mathcal{E}xp(\mu), \quad q_{1,0} = \mu$

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$



Transition rate diagram.

② $PP(\lambda)$ $\qquad S = \{0, 1, 2, \dots\}$

$$T_{i,i+1} \sim \mathrm{Exp}(\lambda), \quad \nu_i = \lambda \quad \forall i$$

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & & \end{array} \\ \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & \cdots & 0 \\ 0 & -\lambda & \lambda & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \end{bmatrix} \end{array}$$

$$0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} \cdots$$

Rate diagram

Lecture - 28

Some more examples.

① Pure - birth process (generalization of $PP(\lambda)$)

$$0 \xrightarrow{\lambda_0} 1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} \cdots$$

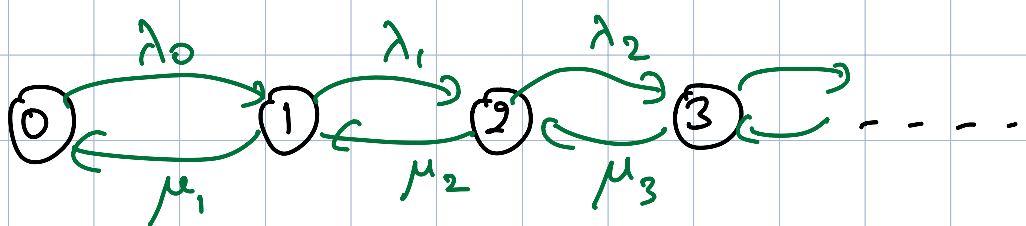
$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & & & \\ & -\lambda_1 & \lambda_1 & & \\ & & -\lambda_2 & \lambda_2 & \\ & & & \ddots & \ddots \end{bmatrix}$$

② Pure death process

$$0 \xleftarrow{\mu_1} 1 \xleftarrow{\mu_2} 2 \xleftarrow{\mu_3} 3 \xleftarrow{\mu_4} \cdots$$

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{bmatrix} 0 & & & \\ \mu_1 & -\mu_1 & & \\ & \mu_2 & -\mu_2 & \\ & & & \ddots \end{bmatrix} \end{array}$$

③ Birth and death process



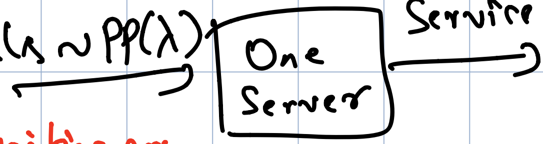
$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{bmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

Suppose CTMC is in state 1.

It spends $\exp(\lambda_1 + \mu_1)$ & transitions to state 0 w.p. $\dfrac{\mu_1}{\lambda_1+\mu_1}$ & to state 2 w.p. $\dfrac{\lambda_1}{\lambda_1+\mu_1}$

(4) Single-server queue (M/M/1)

arrivals $\sim PP(\lambda)$



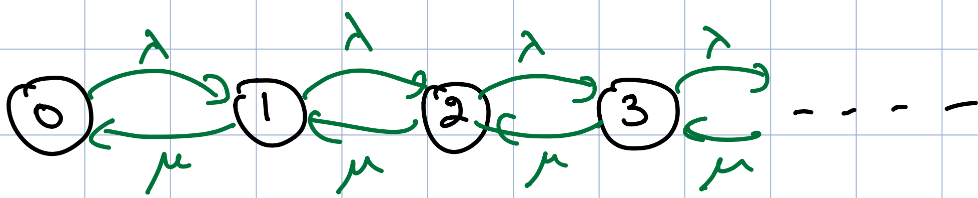
Service time $\sim \exp(\mu)$

large waiting area

(= no arriving customer leaves)

State = # people in the system (= those in service + those waiting)

$S = \{0, 1, 2, \ldots\}$

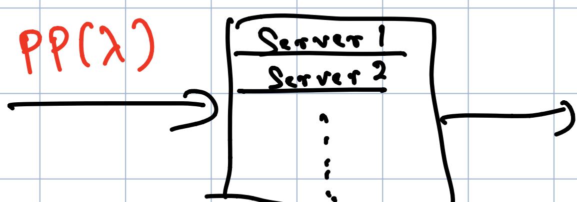


So, we have a birth-death process with $\lambda_i = \lambda$ & $\mu_i = \mu$ $\forall i$

---

(5) Infinite server queue:

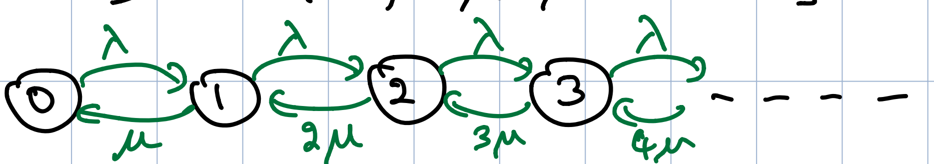
arrivals $\sim PP(\lambda)$



infinite servers, Service time $\sim \exp(\mu)$, $\forall$ servers iid

State = # people in the system.

$S = \{0, 1, 2, \ldots\}$



So, we have a birth-death process with

$$\lambda_i = \lambda \quad \& \quad \mu_i = i\mu, \quad \forall i$$

Lecture-29

(6) Linear-growth model

Colony of individuals

lifetime $\sim \exp(\mu)$

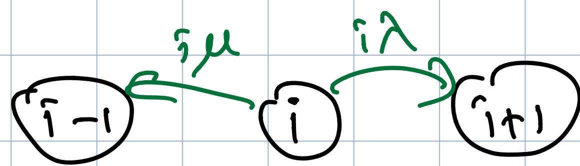
When alive, an individual gives birth according to $PP(\lambda)$, independent of others.

$$S = \{0, 1, 2, \ldots\}$$

State = # individuals in the colony.

State $i \longrightarrow i+1$

a birth by one of "i" individuals



State $i \longrightarrow i-1$

a death of one of "i" individuals

$$q_{i,i+1} = i\lambda$$

$$q_{i,i-1} = i\mu$$

So, this is a birth death process, with

$$\lambda_i = i\lambda, \quad \mu_i = i\mu, \quad \forall i$$

Transient behaviour of CTMCs:

CTMC with initial distribution "a" & generator matrix Q on $S=\{0,1,2,\dots\}$

$$p_j(t) = P(X(t)=j)$$

$$= \sum_{i\in S} P(X(t)=j \mid X(0)=i)\, a_i$$

$$p_j(t) = \sum_{i\in S} a_i\, P_{i,j}(t)$$

Letting $p(t) = [p_j(t)]_{j\in S}$, we have

$$p(t) = a\, P(t),$$

where $P(t) = [P_{i,j}(t)]_{i,j\in S}$, $\quad P_{i,j}(t)\geq 0$, $\sum_j P_{i,j}(t)=1$

Chapman-Kolmogorov equations:

$$P_{i,j}(s+t) = \sum_{k\in S} P_{i,k}(s)\, P_{k,j}(t), \quad \forall\, i,j\in S,\ s,t\geq 0$$

In matrix notation, $\quad P(s+t) = P(s)\, P(t)$

Also, $\quad P(s+t) = P(t)\, P(s)$

i.e., $P(s)\,P(t) = P(t)\,P(s)$, $\forall s,t \geq 0$

These matrices commute

Pf:

$$P_{i,j}(s+t) = P(X(s+t)=j \mid X(0)=i)$$

$$= \sum_{k\in S} P(X(s+t)=j \mid X(0)=i, X(s)=k)\, P(X(s)=k \mid X(0)=i)$$

Markov property

$$\overset{\downarrow}{=} \sum_{k\in S} P(X(s+t)=j \mid X(s)=k)\, P_{i,k}(s)$$

Time-homogeneity

$$\overset{\downarrow}{=} \sum_{k\in S} P(X(t)=j \mid X(0)=k)\, P_{i,k}(s)$$

$$= \sum_{k\in S} P_{i,k}(s)\, P_{k,j}(t). \qquad \blacksquare$$

Forward and backward equations:

CTMC with t.p.m. $P(t)$ and generator matrix Q on $S=\{0,1,2,\dots\}$

Claim: $P(t)$ is differentiable w.r.t. t & satisfies

Backward equation : $\dfrac{d}{dt}P(t) = P'(t) = Q\,P(t)$, $t \geq 0$

Forward equation : $\dfrac{d}{dt}P(t) = P'(t) = P(t)\,Q$, $t \geq 0$, with $P(0) = I$.

Pf: <Claim within claim>

$$P_{i,j}(h) = \delta_{i,j} + q_{i,j} h + o(h),$$

where $\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$

$o(h)$ is a function satisfying $\lim_{h \to 0} \frac{o(h)}{h} = 0$

<Pf within Pf>

question is if starting in i at 0, will the CTMC remain in i throughout or move onto a state $j \neq i$.

Let $N(h) = \#$ transitions in $[0,h]$

Case 1: Prob of staying in i through $[0,h]$

$$P(N(h) = 0 \mid X_0 = i)$$

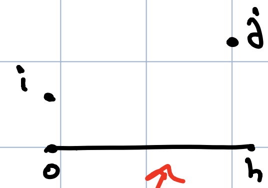
$$= P(\text{staying in } i \text{ during } [0,h] \mid X_0 = i)$$

$$= P(Y_1 > h \mid X_0 = i)$$

$$= e^{-q_i h} = 1 - q_i h + o(h)$$ ← Taylor expansion for exponential

$$= 1 + q_{i,i} h + o(h)$$ (Recall $q_i = \sum_j q_{i,j}$ & $q_{i,i} = -q_i$)

Case 2: Move out of i & stay in that state upto h



$$P(N(h)=1 \mid X_0=i)$$

$$= P(Y_1 \le h,\ Y_1+Y_2 > h \mid X_0=i)$$

if only one transition occurs, then holding time of i $\le h$ & the second transition occurs after h, i.e., $Y_1+Y_2 > h$

$$= \sum_{\substack{j\in S\\ j\neq i}} \int_0^h \left(q_i e^{-q_i s}\right) p_{i,j}\, e^{-q_j(h-s)}\, ds$$

$$= \sum_{\substack{j\in S\\ j\neq i}} q_i\, p_{i,j}\, e^{-q_j h} \left(\frac{1-e^{-(q_i-q_j)h}}{q_i-q_j}\right) \quad (\text{assuming } q_i \neq q_j)$$

$$= q_i h + o(h)$$ ← Case $q_i = q_j$ leads to this as well.

Case 3: 2 or more transitions occur in $[0,h]$

It can be shown that

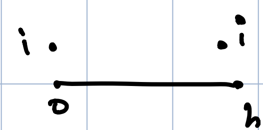
$$P(N(h)\geq 2 \mid X_0=i) = o(h)$$

Summary: ① $P(N(h)=0 \mid X(0)=i) = 1+q_{i,i}h+o(h)$

② $P(N(h)=1 \mid X(0)=i) = q_i h + o(h)$

③ $P(N(h)\geq 2 \mid X(0)=i) = o(h)$

Using these, we get for $j=i$



$$P_{i,i}(h) = P(X(h)=i \mid X(0)=i)$$

$$= P(X(h)=i \mid X(0)=i, N(h)=0)\ P(N(h)=0 \mid X(0)=i)$$

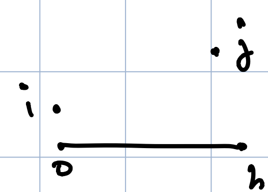
$$+ P(X(h)=i \mid X(0)=i, N(h)=1)\, P(N(h)=1 \mid X(0)=i)$$

$$+ P(X(h)=i \mid X(0)=i, N(h)\geq 2)\, P(N(h)\geq 2 \mid X(0)=i)$$

$$= 1 \times (1 + q_{i,i} h + o(h)) + 0 \times (q_i h + o(h)) + o(h)$$

$$P_{i,i}(h) = 1 + q_{i,i} h + o(h) \quad \text{— } (*)$$

For $j \neq i$,



$$P_{i,j}(h) = P(X(h)=j \mid X(0)=i)$$

$$= P(X(h)=j \mid X(0)=i, N(h)=0)\, P(N(h)=0 \mid X(0)=i)$$

$$+ P(X(h)=j \mid X(0)=i, N(h)=1)\, P(N(h)=1 \mid X(0)=i)$$

$$+ P(X(h)=j \mid X(0)=i, N(h)\geq 2)\, P(N(h)\geq 2 \mid X(0)=i)$$

$$= 0 \times (1 + q_{i,i} h + o(h)) + P(X_1=j \mid X_0=i)(q_i h + o(h)) + o(h)$$

$$= P_{i,j}(q_i h + o(h))$$

$$P_{i,j}(h) = q_{i,j} h + o(h). \quad \text{— } (**)$$

$(*)$ & $(**)$ imply the (claim within claim)

Pf of backward equation:

→ Chapman-Kolmogorov eqns.

$$P_{i,j}(t+h) = \sum_{k\in S} P_{i,k}(h)\, P_{k,j}(t) \quad \text{———} (***)$$

$$P_{i,j}(t+h) - P_{i,j}(t) = \sum_{k\in S} (\delta_{i,k} + q_{i,k} h + o(h))\, P_{k,j}(t)$$

$$\frac{P_{i,j}(t+h) - P_{i,j}(t)}{h} = \sum_{k\in S} \frac{(\delta_{i,k} + q_{i,k} h + o(h))\, P_{k,j}(t)}{h}$$

$$\frac{P_{i,j}(t+h) - P_{i,j}(t)}{h} = \sum_{k\in S} q_{i,k}\, P_{k,j}(t) + \frac{o(h)}{h}$$

Taking limit as $h\to 0$, we get

$$P_{i,j}'(t) = \sum_{k\in S} q_{i,k}\, P_{k,j}(t)$$

or, $P'(t) = Q\, P(t)$

Interchange h & t in $(***)$ & repeat the steps to obtain $P'(t) = P(t)\, Q$

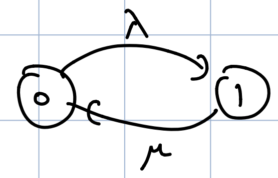
or $P_{i,j}'(t) = \sum_{k\in S} P_{i,k}(t)\, q_{k,j}$

$\blacksquare$

Lecture-30

Example: Two-state machine (see example above)

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$



Backward equations: $p'_{i,j}(t) = \sum_{k \in S} q_{i,k}\, p_{k,j}(t)$

Forward eqns: $p'_{i,j}(t) = \sum_{k} p_{i,k}(t)\, q_{k,j}$

Initial condition: $P(0) = I$.

Forward equations

$$p'_{0,0}(t) = -\lambda p_{0,0}(t) + \mu p_{0,1}(t) \qquad p_{0,0}(0) = 1$$

$$p'_{0,1}(t) = \lambda p_{0,0}(t) - \mu p_{0,1}(t) \qquad p_{0,1}(0) = 0$$

$$p'_{1,0}(t) = -\lambda p_{1,0}(t) + \mu p_{1,1}(t) \qquad p_{1,0}(0) = 0$$

$$p'_{1,1}(t) = \lambda p_{1,0}(t) - \mu p_{1,1}(t) \qquad p_{1,1}(0) = 1$$

Take the first two forward equation & add

$$p_{0,0}(t) + p_{0,1}(t) = 1.$$

So,
$$p'_{0,0}(t) = -\lambda p_{0,0}(t) + \mu p_{0,1}(t)$$

$$= -\lambda p_{0,0}(t) + \mu(1 - p_{0,0}(t))$$

$$p'_{0,0}(t) = -(\lambda+\mu) p_{0,0}(t) + \mu$$

Let
$$h(t) = p_{0,0}(t) - \frac{\mu}{(\lambda+\mu)} \qquad \text{—— } (*)$$

$$h'(t) = \mu - (\mu+\lambda)\left(h(t) + \frac{\mu}{\mu+\lambda}\right)$$

$$= -(\mu+\lambda)\, h(t)$$

$$\frac{h'(t)}{h(t)} = -(\mu+\lambda)$$

Integrate on both sides

$$\log h(t) = -(\mu+\lambda)t + C$$

$$h(t) = K \exp(-(\mu+\lambda)t)$$

Using this in (*), we get

$$P_{0,0}(t) = K \exp(-(\mu+\lambda)t) + \frac{\mu}{\lambda+\mu}$$

Using $P_{0,0}(0) = 1$, we get $K = \dfrac{\lambda}{\lambda+\mu}$

So, $$P_{0,0}(t) = \frac{\lambda}{\lambda+\mu} \exp(-(\mu+\lambda)t) + \frac{\mu}{\lambda+\mu}.$$

$$P_{0,1}(t) = 1 - P_{0,0}(t) = \boxed{\text{H.W.}}$$

Use a similar technique to obtain expressions for $P_{1,0}(t)$ & $P_{1,1}(t)$

Forward equations for a pure-birth process

$$0 \xrightarrow{\lambda_0} 1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} \cdots$$

Claim: $P_{i,i}(t) = \exp(-\lambda_i t), \quad \forall i \geq 0$

$$P_{i,j}(t) = \lambda_{j-1} \exp(-\lambda_j t) \int_0^t \exp(\lambda_j s)\, P_{i,j-1}(s)\, ds, \quad j \geq i+1$$

Pf: Forward equations:

$$P_{i,i}'(t) = -\lambda_i P_{i,i}(t) \qquad \text{--- } (*)$$

$$P_{i,j}'(t) = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{i,j}(t) \qquad \text{--- } (**)$$

from $(*)$, $\dfrac{P_{i,i}'(t)}{P_{i,i}(t)} = -\lambda_i$

leading to $P_{i,i}(t) = \exp(-\lambda_i t)$

From $(**)$,

$$\exp(\lambda_j t)\left(P_{i,j}'(t) + \lambda_j P_{i,j}(t)\right) = \left(\lambda_{j-1} P_{i,j-1}(t)\right) \exp(\lambda_j t)$$

$$\Updownarrow$$

$$\frac{d}{dt}\left(\exp(\lambda_j t) P_{i,j}(t)\right) = \left(\lambda_{j-1} P_{i,j-1}(t)\right) \exp(\lambda_j t)$$

Integrate on both sides & use $P_{i,j}(0) = 0$, to obtain

$$p_{i,j}(t) = \lambda_{j-1} \exp(-\lambda_j t) \int_0^t \exp(\lambda_j s)\, p_{i,j-1}(s)\, ds, \quad j \geq i+1$$

Occupancy times

$V_j(t) =$ amount of time spent in "j" over $(0,t]$

(Note: $V_j(0) = 0 \ \forall j$)

$$M_{i,j}(t) = E\left(V_j(t) \mid X(0)=i\right), \quad \forall i, j, \ t \geq 0$$

$$M(t) = \left[M_{i,j}(t)\right]_{i,j \in S}$$

Claim:

$$M(t) = \int_0^t P(u)\, du, \quad t \geq 0.$$

Pf: Fix $j \in S$. Let $Z(u) = \begin{cases} 1 & \text{if } X(u) = j \\ 0 & \text{else} \end{cases}$

$$V_j(t) = \int_0^t Z(u)\, du$$

$$M_{i,j}(t) = E\left(V_j(t) \mid X(0)=i\right)$$

$$= E\left(\int_0^t Z(u)\, du \mid X(0)=i\right)$$

$$= \int_0^t E\left(Z(u) \mid X(0)=i\right) du$$

$$= \int_0^t P\left(X(u)=j \mid X(0)=i\right) du$$

$$= \int_0^t P_{i,j}(u)\, du$$

$\blacksquare$

H.W. Calculate occupancy times for the 2-state examples.

Calculation of $P(t)$: Finite state space.

Exponential of a matrix

$$e^A := I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$$ for any square matrix A.

Note! ① If $A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_N \end{bmatrix}$, then

$$e^A = \begin{bmatrix} e^{a_1} & & 0 \\ & \ddots & \\ 0 & & e^{a_N} \end{bmatrix}$$

② $e^{0} = I$ zero-matrix.

Connection to CTMC:

Claim: For a CTMC with generator Q & a finite state space,

$$P(t) = e^{Qt}, \quad t \geq 0$$

Pf:

$$e^{Qt} = I + \sum_{n=1}^{\infty} \frac{(Qt)^n}{n!}$$

$$\frac{d}{dt} e^{Qt} = Q e^{Qt} = e^{Qt} Q \quad \leftarrow \text{check this}$$

$$\& \quad e^0 = I.$$

Recall forward/backward equations: $P'(t) = Q P(t) = P(t) Q$ with $P(0) = I$.

& the solution is unique.

So, $P(t) = e^{Qt}$.

Another claim: Suppose Q is diagonalizable, i.e.,

$$Q = X D X^{-1}, \quad (D \to \text{diagonal})$$

Then, $P(t) = X e^{Dt} X^{-1}$

Pf:

$$e^{Qt} = I + \sum_{n=1}^{\infty} \frac{(X D X^{-1} t)^n}{n!}$$

$$= I + \sum_{n=1}^{\infty} \frac{X (Dt)^n X^{-1}}{n!}$$

$$= X\left(\underbrace{I + \sum_{n=1}^{\infty} \frac{(Dt)^n}{n!}}\right) X^{-1} \longrightarrow e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_N t} \end{bmatrix}$$

Note: If $D = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}$

$$X = \begin{bmatrix} | & & & | \\ x_1 & — & -- & x_N \\ | & & & | \end{bmatrix} \quad Y = \begin{bmatrix} — & y_1 & — \\ & \vdots & \\ — & y_N & — \end{bmatrix}$$

Then, $$X e^{Dt} X^{-1} = \sum_{i=1}^{N} e^{\lambda_i t} x_i y_i$$

Lecture 31

Two state CTMC :-

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$Q = X D X^{-1}$, where

$$X = \begin{bmatrix} 1 & \lambda/\lambda+\mu \\ 1 & -\mu/\lambda+\mu \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & -(\lambda+\mu) \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} \mu/\lambda+\mu & \lambda/\lambda+\mu \\ 1 & -1 \end{bmatrix}$$

H.W.: Check this

$$P(t) = e^{Qt} = X \begin{bmatrix} 1 & 0 \\ 0 & e^{-(\lambda+\mu)t} \end{bmatrix} X^{-1}$$

H.W. Check if the expression above tallies with the one obtained by solving the backward equations earlier.

For DTMCs, we looked at the eigenvalues of P.
In a CTMC, we do the same for Q.

Claim: CTMC with generator Q, which has eigenvalues $\lambda_1, \ldots, \lambda_N$.

1. At least one of the eigenvalues is zero.
2. Let $q = \max\{-q_{i,i},\ i=1..N\}$

Then, $|\lambda_i + q| \le q$ for $i=1..N$.

Pf: Let $P = I + \frac{1}{q} Q$

Is P stochastic? Yes, since row-sums of $P=1$
& P is composed of non-negative entries.

(*) For a stochastic matrix, at least one of the eigenvalues is 1.

Eigenvalues of $P = \left\{ 1 + \frac{\lambda_i}{q},\ i=1..N \right\}$

Using *, $1 + \frac{\lambda_i}{q} = 1$ for at least one i

$\Rightarrow$ At least one of the λ_is is zero.

(**) $|\beta| \leq 1$ for a eigenvalue β of P.

Using (**), $$\left|1 + \frac{\lambda_i}{q}\right| \leq 1$$

$\blacksquare$

A brief tour of Laplace transforms:

For a $f : [0,\infty) \to (-\infty, \infty)$, the Laplace transform (LT) is defined as

$$f^*(s) = \int_0^\infty e^{-sx} f(x)\, dx$$

Properties:- (See Appendix F of Kulkarni's book for the full list)

① $LT(af + bg) = a f^*(s) + b g^*(s)$

② $LT(f'(t))$

$$= \int_0^\infty e^{-sx} f'(x)\, dx$$

$$= e^{-sx} f(x)\Big]_0^\infty + s \int_0^\infty e^{-sx} f(x)\, dx$$

$$= -f(0) + s f^*(s)$$

< End of the tour >

LT in CTMCs:

Define LT of $P_{i,j}(t)$ as

$$P^*_{i,j}(s) = \int_0^\infty e^{-st} P_{i,j}(t)\, dt, \quad \text{for } Re(s) > 0$$

real part of

$$P^*(s) = [P^*_{i,j}(s)]_{i,j \in S}$$

Claim: $\quad P^*(s) = (sI - Q)^{-1}, \quad Re(s) > 0$

Pf:

$$\int_0^\infty e^{-st} P'_{i,j}(t)\, dt = s\, P^*_{i,j}(s) - P_{i,j}(0)$$

Recall from forward/backward equations, we have

$$P'(t) = Q\, P(t)$$

LT on both sides: $\quad sP^*(s) - I = Q\, P^*(s) \quad \text{—} (*)$

Similarly, $\quad P'(t) = P(t)\, Q$

LT on both sides: $\quad sP^*(s) - I = P^*(s)\, Q \quad \text{—} (**)$

From $(*)$, $\quad (sI - Q)\, P^*(s) = I$

From $(**)$, $\quad P^*(s)\,(sI - Q) = I$

$\Rightarrow (sI - Q)$ is invertible

$$P^*(s) = (sI - Q)^{-1}. \quad \blacksquare$$

Example: Two-state CTMC

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$p^*(s) = (sI - Q)^{-1}$$

$$p^*(s) = \frac{1}{s(s+\lambda+\mu)} \begin{bmatrix} s+\mu & \lambda \\ \mu & s+\lambda \end{bmatrix}$$ ← Check this (h.w.)

$$p^*_{0,0}(s) = \frac{s+\mu}{s(s+\lambda+\mu)}$$

$$= \frac{\mu}{\lambda+\mu} \cdot \frac{1}{s} + \frac{\lambda}{\lambda+\mu} \cdot \frac{1}{s+\lambda+\mu}$$

Going from $p^*_{0,0}$ to $p_{0,0}$ using Appendix F of textbook,

$$p_{0,0}(t) = \underbrace{\frac{\mu}{\lambda+\mu} \cdot 1 + \frac{\lambda}{\lambda+\mu} \cdot \exp(-(\lambda+\mu)t)}$$

① Check if this tallies with the solutions obtained using forward equation & diagonalization.

② Find $p_{0,1}(t)$, $p_{1,0}(t)$, $p_{1,1}(t)$ using LT.

Lecture - 32

Example: Poisson process $PP(\lambda)$

let $p_i(t) = p_{0,i}(t) = P(X(t)=i \mid X(0)=0)$

Forward equations:

$$p_0'(t) = -\lambda p_0(t) \quad \text{--- (1)}$$

For $i \geq 1$,

$$p_i'(t) = -\lambda p_i(t) + \lambda p_{i-1}(t) \quad \text{--- (2)}$$

Initial conditions: $p_0(0) = 1$, $p_i(0) = 0$, $i \geq 1$

LT on both sides of (1) & (2):

$$s p_0^*(s) - 1 = -\lambda p_0^*(s) \quad \text{--- (3)}$$

$$s p_i^*(s) = -\lambda p_i^*(s) + \lambda p_{i-1}^*(s) \quad \text{--- (4)}$$

(3) $\Rightarrow$

$$p_0^*(s) = \frac{1}{\lambda + s}$$

(4) $\Rightarrow$

$$p_i^*(s) = \frac{\lambda}{\lambda+s} p_{i-1}^*(s)$$

$$\vdots$$

$$p_i^*(s) = \left(\frac{\lambda}{\lambda+s}\right)^i p_0^*(s)$$

$$p_i^*(s) = \frac{\lambda^i}{(\lambda+s)^{i+1}}$$

Inverting (using Appendix F of the book), we obtain

$$p_i(t) = \lambda^i \frac{e^{-\lambda t} t^i}{i!}$$

i.e., $$p_i(t) = \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$ $\leftarrow$ Poisson (λt) distribution.

Absorption probabilities:

$$T = \min\{t \geq 0 \mid X(t) = 0\}$$

$$v_i = P(T = \infty \mid X(0) = i), \quad i \geq 1$$

Let B be a sub-matrix of the generator Q, obtained by deleting the row & column of state 0.

Claim: Let $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \end{bmatrix}$.

Then, v is the largest solution, bounded above by 1, to

$$Bv = 0$$

Pf: Let $\{X_n\}$ be the embedded DTMC.

$v_i = P($ CTMC $\{X(t)\}$ never visits $0 \mid X(0) = i)$

$v_i = P($ DTMC $\{X_n\}$ never visits $0 \mid X_0 = i)$

From DTMC theory covered earlier

v_i satisfies $v = \tilde{B} v$, where $\tilde{B}$ is a submatrix of t.p.m. of DTMC, obtained by deleting row/col of state 0.

$$v_i = \sum_{j=1}^{\infty} P_{i,j} v_j \quad , \quad i \geq 1$$

$P_{i,j}$: transition probabilities of the embedded DTMC

$$v_i = \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \left(\frac{q_{i,j}}{q_i} \right) v_j$$

$$v_i q_i = \sum_{j \neq i} q_{i,j} v_j$$

$$-q_{i,i} v_i = \sum_{j \neq i} q_{i,j} v_j$$

$$\sum_j q_{i,j} v_j = 0 \iff B v = 0$$

Proof of maximality: Follows in a similar manner as in the proof for DTMC (i.e., $v = \tilde{B} v$).

$\blacksquare$

First moment of T:

$$m_i = E(T \mid X(0) = i) \, , \quad i \geq 1$$

Note: $m_i = \infty$ if $v_i > 0$.

So, we focus on $v_i = 0$ case.

Claim: Suppose $\vartheta = 0$.

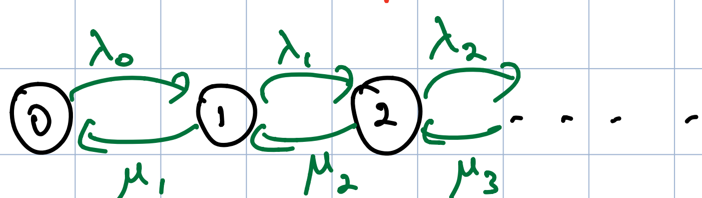
Then, $m = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \end{bmatrix}$ is the smallest non-negative solution to

$$Bm + \mathbb{1} = 0$$

$\mathbb{1}$ → vector of ones.

Pf: reading exercise. ∎

Example: Birth-death process.



$$T = \min\{t \geq 0 \mid X(t) = 0\}$$

To find ϑ, solve $\sum_j q_{i,j}\vartheta_j = 0$, $i \geq 1$

$$\lambda_i \vartheta_{i+1} + \mu_i \vartheta_{i-1} - (\lambda_i + \mu_i)\vartheta_i = 0, \quad i \geq 1$$

Boundary condition: $\vartheta_0 = 0$

$$\vartheta_i = \left(\frac{\mu_i}{\lambda_i + \mu_i}\right)\vartheta_{i-1} + \left(\frac{\lambda_i}{\lambda_i + \mu_i}\right)\vartheta_{i+1}$$

This is in the form of eq. (*) on p. 32 of DTMCs-transient behaviour, where a general random walk is analyzed.

Using the DTMC analysis, we have

Let $p_i = \dfrac{\lambda_i}{\lambda_i+\mu_i}$, $q_i = \dfrac{\mu_i}{\lambda_i+\mu_i}$

$$v_i = \begin{cases} \sum_{j=0}^{i} \alpha_j \Big/ \sum_{j=0}^{\infty} \alpha_j & , \text{ if } \sum_j \alpha_j < \infty \\ 0 & \text{else,} \end{cases}$$

where $\alpha_0 = 1$, $\alpha_i = \dfrac{\mu_1 \cdots \mu_i}{\lambda_1 \cdots \lambda_i}$, $i \geq 1$

To find m, assume $\sum_j \alpha_j = \infty$

$Bm + \mathbb{1} = 0$ can be written as

$$\lambda_i m_{i+1} + \mu_i m_{i-1} - (\lambda_i+\mu_i) m_i + 1 = 0, \quad i \geq 1$$

$$m_0 = 0$$

$$\Updownarrow$$

$$m_i = \frac{1}{\lambda_i+\mu_i} + \frac{\mu_i}{\lambda_i+\mu_i} m_{i-1} + \frac{\lambda_i}{\lambda_i+\mu_i} m_{i+1}$$

Following the technique in example 2 on p. 37 of "DTMCs-transient behaviour", we obtain

$$m_i = \sum_{k=0}^{i-1} \alpha_k \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j \alpha_j}$$

Limiting behaviour of CTMCs

In the limit as $t \to \infty$, what happens to

(i) $P(t)$

(ii) $\frac{M(t)}{t}$, where $M(t) = \int_0^t P(u)\,du$

Two-state CTMC:

$$P(t) = e^{Qt} = X \begin{bmatrix} 1 & 0 \\ 0 & e^{-(\lambda+\mu)t} \end{bmatrix} X^{-1}, \text{ where } X = \begin{bmatrix} 1 & \lambda/\lambda+\mu \\ 1 & -\mu/\lambda+\mu \end{bmatrix}$$

$$\lim_{t\to\infty} P(t) = \begin{bmatrix} \mu/\lambda+\mu & \lambda/\lambda+\mu \\ \mu/\lambda+\mu & \lambda/\lambda+\mu \end{bmatrix}$$

H.W. Calculate $M(t)$ using $M(t) = \int_0^t P(u)\,du$

& show that $\lim_{t\to\infty} \frac{M(t)}{t} = \begin{bmatrix} \mu/\lambda+\mu & \lambda/\lambda+\mu \\ \mu/\lambda+\mu & \lambda/\lambda+\mu \end{bmatrix}$

Remark: the limits of $P(t)$ & $\frac{M(t)}{t}$ do not depend on the initial distribution.

Lecture-33

Stationary distribution

Ref: Kulkarni's book.

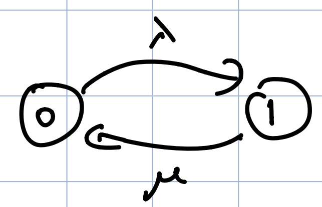
But, for the proofs, consult Norris "Markov chains" (See Sec 3.5 there)

Communicating class:

A set $C \subseteq S$ in a CTMC is a (closed) communicating class if C is a (closed) communicating class of the underlying DTMC.

Irreducibility: CTMC is irreducible if the underlying DTMC is irreducible.

Two-state CTMC:

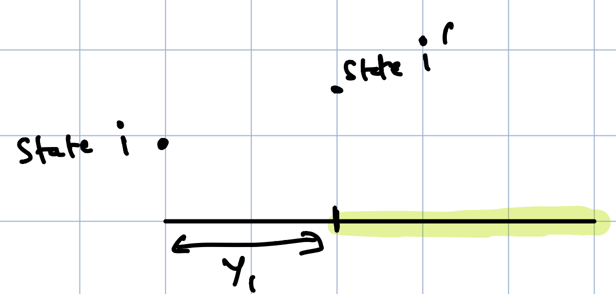


Irreducible if $\lambda, \mu > 0$

Transience & recurrence

First-passage time $T_i = \inf\{ t \geq Y_1 \mid X(t) = i \}, \ \forall i$

first sojourn time.



Remark: T_i is well-defined if $Y_1 < \infty$ w.p. 1 $\iff q_i > 0$

If $q_i = 0$ then i is absorbing.

Let $u_i = P(T_i < \infty \mid X(0) = i)$ $\Leftarrow$ Prob. of returning to i

$m_i = E(T_i \mid X(0) = i)$ $\Leftarrow$ mean return time

Note: $u_i < 1 \Rightarrow m_i = \infty$

A state i with $q_i > 0$ is $\begin{cases} \text{transient} & \text{if } u_i < 1 \\ \text{recurrent} & \text{if } u_i = 1 \end{cases}$

Note: If $q_i = 0$, then i is recurrent.

Claim: A state "i" is recurrent (transient) in a CTMC iff "i" is recurrent (transient) in the underlying DTMC.

Pf: Straightforward ■

A state i with $q_i > 0$ is $\begin{cases} \text{null recurrent} & \text{if } m_i = \infty \\ \text{positive recurrent} & \text{if } m_i < \infty \end{cases}$

Note: If $q_i = 0$, i is positive recurrent.

Remark: So, as in a DTMC, transience/recurrence are class properties.

Stationary distribution:

A vector π is a stationary distribution of a CTMC if (i) $\pi_j \geq 0, \forall j$ (ii) $\sum_j \pi_j = 1$

(iii) $\pi = \pi P(t), \forall t \geq 0$

Remark: If $X(0)$ has distribution π, then $X(t)$ has distribution π as well.

Claim: $\pi = \pi P(t), t \geq 0 \iff \pi Q = 0$

Pf:

$$
\begin{aligned}
\pi Q = 0 &\iff \pi Q^n = 0 \quad \forall n \geq 1 \\
&\iff \sum_{n=1}^{\infty} \frac{t^n}{n!} \pi Q^n = 0, \quad \forall t \\
&\iff \pi \left(\sum_{n=1}^{\infty} \frac{t^n Q^n}{n!} \right) = 0 \\
&\iff \pi \left(\sum_{n=0}^{\infty} \frac{t^n Q^n}{n!} \right) = \pi \\
&\iff \pi e^{Qt} = \pi \\
&\iff \pi P(t) = \pi \quad (\text{since } P(t) = e^{Qt})
\end{aligned}
$$

$\blacksquare$

So, π is stationary if it satisfies $\pi Q = 0$.

Limiting behaviour : Transient case

Claim: $\{X(t), t \geq 0\}$ irreducible transient CTMC.

$$\lim_{t \to \infty} P_{i,j}(t) = 0 \quad , \forall i,j \in S$$

Pf: Let $\{X_n\}$ be the underlying DTMC.
This DTMC is irreducible + transient.

$E($ # visits to j starting in i over the infinite horizon$)$
$< \infty$

Each visit to j, the CTMC spends $\frac{1}{q_j}$ time in expectation

So, $E($ time spent by CTMC in j over $(0,\infty)$, starting in $i)$
$< \infty$

$$\lim_{t \to \infty} \underbrace{M_{i,j}(t)}_{} < \infty$$

this is the expectation of time spent in j starting in i over $(0,t)$

$$\lim_{t \to \infty} M_{i,j}(t) = \lim_{t \to \infty} \int_0^t P_{i,j}(t)\, dt < \infty$$

$$P_{i,j}(t) \geq 0.$$

So, $$\lim_{t \to \infty} P_{i,j}(t) = 0.$$ $\blacksquare$

Limiting behaviour : Null-recurrent case

Claim: $\{X(t), t \geq 0\}$ irreducible null-recurrent CTMC.

$$\lim_{t \to \infty} P_{i,j}(t) = 0 \quad , \forall i,j \in S$$

Pf: Skipped. $\blacksquare$

Limiting behaviour : Positive-recurrent case

Suppose π is the stationary distribution of the CTMC &
λ is the stationary distribution of the underlying DTMC.

Then, we can relate them by

$$\lambda_i = \pi_i v_i \quad , \forall i$$

To see this, $$\lambda_i = \sum_j P_{j,i} \lambda_j$$

Using $\lambda_i = \pi_i v_i$, we have $$\pi_i v_i = \sum_j P_{j,i} v_j \pi_j$$

Using $P_{j,i} = \dfrac{q_{j,i}}{q_j}$, we have $\quad \pi_i q_i = \sum_j q_{j,i} \pi_j$

Using $q_i = -q_{i,i}$ $\quad 0 = \sum_{j \neq i} q_{j,i} \pi_j + q_{i,i} \pi_i$

Or, $\quad \sum_j q_{j,i} \pi_j = 0 \iff \pi Q = 0$

Big Facts:

① Consider an irreducible CTMC. Then,

it is positive recurrent

if and only if

there exists a stationary distribution π

(i.e., $\pi Q = 0$ & $\sum \pi_j = 1$, $\pi_j \geq 0$)

② For an irreducible, positive recurrent CTMC

(i) $\lim_{t \to \infty} P_{i,j}(t) = \pi_j$, $\quad \forall i, j \in S$

(ii) $\lim_{t \to \infty} \dfrac{M_{i,j}(t)}{t} = \pi_j$, $\quad \forall i, j \in S$

Pf: Check Norris' textbook on Markov chains, in particular, Sections 3.5 & 3.6 there. ■

$\pi Q = 0$ is called "Balance equations".

Why?

$$\sum_{j \neq i} \pi_j q_{j,i} + \pi_i q_{i,i} = 0$$

$$-\pi_i q_{i,i} = \sum_{j \neq i} \pi_j q_{j,i}$$

$$\sum_{j \neq i} \pi_i q_{i,j} = \sum_{j \neq i} \pi_j q_{j,i}$$

→ rate at which system enters state i

$\pi_i q_i$ = rate at which the system leaves state i

Balance equations: two rates are equal in steady state.

<u>Example:</u> Two-state CTMC

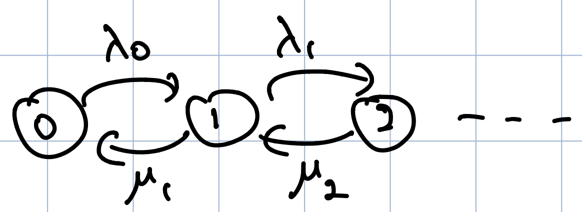
$$0 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 1$$

$$\lambda \pi_0 = \mu \pi_1$$

$$\pi_0 + \pi_1 = 1$$

$$\pi_0 = \frac{\mu}{\lambda + \mu}, \quad \pi_1 = \frac{\lambda}{\lambda + \mu}$$

Example: Birth-death process



$$\lambda_0 \pi_0 = \mu_1 \pi_1$$

$$(\lambda_i + \mu_i)\pi_i = \lambda_{i-1}\pi_{i-1} + \mu_{i+1}\pi_{i+1}$$