Homogeneous Hierarchical Composition of Areas in Multi-robot Area Coverage

Sriram Raghavan[1] and Ravindran B[2]

[1] Intel India
sriram.raghavan@intel.com
[2] Department of Computer Science & Engineering, IIT Madras
ravi@cs.iitm.ernet.in

Abstract. Multi-robot area coverage poses several research challenges. The challenge of coordinating multiple robots' actions coupled with the challenge of minimizing the overlap in coverage across robots becomes even more complex and critical when large teams and large areas are involved. In fact, the efficiency critically hinges on the coordination algorithms used and the robot capabilities.

Multi-robot coverage of such large areas can be tackled by the divide-and-conquer policy; decomposing the coverage area into several small coverage grids. It is fairly simple to devise algorithms to minimize the overlap in small grids by making simple assumptions. If the overlap ratio of these small grids can be controlled, one may be able to integrate them appropriately to cover the large grid.

In this paper, we introduce homogeneous hierarchical composition grids to decompose a coverage area into several small coverage primitives with appropriately sized robot teams. These coverage grids are viewed as cells at a Meta level and composed hierarchically with such teams functioning as a single unit. We state and prove an associated theorem that provides very good scaling properties to large grids. We have performed simulated studies to validate the claims and study performance.

1 Introduction

Multi-Robot Area Coverage involves visiting every point within a given area by a team of mobile robots. Such tasks are typical to coordinated tasks such as Robotic vacuuming, Robotic de-mining or Robotic rescue. In such applications, it is sufficient if any one member of the team visits a particular point in the coverage area as repeated visits provides no additional information or value. Revisits are considered as overhead on the task completion. Such coverage tasks are usually characterized by few points for entry and no a priori knowledge about the terrain for the area. However, it is reasonable to assume that periphery can be identified through vision or radio beacons.

Consider a situation where a group of rescue robots form a rescue team to evacuate survivors from a building which is devastated by natural calamity. In such a situation, it is required that the team coordinates its actions so as to rescue as many survivors as

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possible, quickly. We assume, for sake of simplicity, that all searches are restricted to searching on planar space. It is typical of such a scenario to round up the area (periphery of coverage area is known) and scan extensively within. In order to be effective, the team members must spread out after entering the area and avoid revisits to a location. Coordination also necessitates the need to set up a common reference during communication which requires a coordination architecture [1] to be built into these robot team members. This is then achieved thru exchange of messages to decide on future course of action. Since the robots are mobile, the coordination architecture must support one or more wireless communication technologies.

Once these are in place, the robots should use a common protocol or coordination algorithm to exchange their findings and status (robot states) to decide the optimal next step. While it is best if robots synchronized with each other after every step to take the optimal next step, it results in significant communication overhead. The tradeoff between the periodicity of communication and the level of optimality in robot actions is decided based on application needs. If the area is very vast, then the robots are divided into smaller teams and are made to cover smaller regions in parallel. This technique has three distinct advantages, viz., coordination in smaller teams is simpler and faster, the robots can cover the smaller regions using simple and efficient algorithms to minimize overlap and if regions do overlap, it is restricted to the smaller team and not the entire robot group.

Area coverage, in literature, is performed using two kinds of area decomposition: approximate cellular decomposition [4,10,14] where the coverage area is approximated as a grid of cells and exact cellular decomposition [2,7] where it is exactly mapped by one or more such grids. *The cell is the footprint of a robot and the area covered by it in one unit of time*. Increasing the size of the cell implies increase in robot coverage ability in unit time which may be required to support advanced applications.

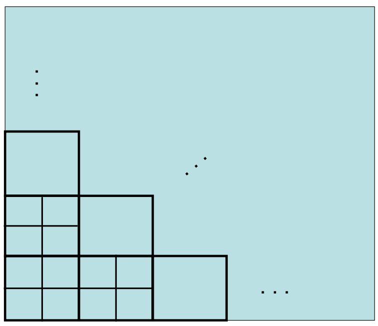

Fig. 1. Illustrating concept of hierarchy by building up using small areas

In this paper, we introduce the concept of teams of robots and coverage primitives which form the building blocks of coverage. These robots are simple in design with limited sensing and computational capabilities being easy to program and cheap to build. However, they cannot use conventional coordination techniques owing to their limited capabilities.This paper briefly discusses various coordination algorithms developed by us to suit different contexts and robot capabilities. These algorithms quantitatively measure the overlap using a parameter called the overlap_ratio and attempt to minimize it using various strategies in the different contexts. A team will employ an appropriate algorithm to cover a primitive and move from one primitive to another after coverage. The algorithm is selected based on the application needs and individual robot capabilities. Since a team of robots move from one primitive to another during coverage, these primitives may be equated to 'large' cells being covered by robots with greater degrees of freedom. We use this concept and compose very large areas using primitives and divide the robots as appropriate directed by application needs. This will give rise to hierarchical composition of the area where we prove that the total overlap_ratio of the coverage area is the sum of overlap_ratios of immediate two sublevels levels and this result is scalable to any number of levels in hierarchy.

The rest of the paper is organized as follows. In section 2, we present the state-of-the-art in multi-robot are coverage. In section 3, we describe the different classes of multi-robot coordination algorithms we have developed that successively reduce the overlap in coverage across robots and also state the assumptions on which these algorithms are based. We also classify these algorithms and categorize the same. In section 4, we present the homogeneous hierarchical composition theorem and discuss the test cases studied. In section 5, we discuss the experimental setup, summarize the results and discuss our findings. In section 6, we conclude by providing a brief summary of the work achieved in this paper.

2 State-of-the-Art in Multi-robot Area Coverage

There are three dimensions in which mobile robot coordination for achieving a coverage task can be classified. They are: *Coverage Problem*, *Coordination Problem*, and *Communication Problem*. Cao et al. [13] provide a classification of multi-agent robotics along the dimensions of communication, computation and other capabilities. Choset surveys the area coverage problem [7] and introduces some *basic coverage heuristics*. Butler et al. [8] describe algorithms that guarantee coverage of rectilinear environments by a team of robots. Rekleitis et al [9] describe a graph-based multi-robot exploration and mapping approach which keeps two robots in closely-coupled coordination each robot is always in line-of-sight of the other.

Solanas and Garcia [2] present an unsupervised clustering algorithm that partitions the unknown space into as many cells as the number of mobile robots. The assignment of regions to the various robots is based on bids that are estimates of information gain traded-off against traveling costs to that region. Simmons et al. [11] describe a centralized exploration and mapping algorithm that uses maximum likelihood to find maps maximally consistent with the sensor data from the region. Zlot et al. [4] present a totally distributed exploration algorithm in mobile robots based on the market economy which minimizes traveling costs and maximizes information gain.

In ant-robot based terrain coverage [6], simple robots with minimal sensory capabilities perform at least once-coverage or continual coverage of an unknown terrain. The terrain is exactly decomposed into cells, each of which is the size of a robot. The work assumes that multiple robots may visit a single cell simultaneously without hindering the coverage path of other robots. Robots move in perfect synchronization during coverage without communicating with one another but rely on pheromone trails left by other robots earlier at that location. The action selection mechanism is based on an arbitrary function used to select the action that minimizes some cost function known a priori. Kube and Bonabeau have also looked at ant-like movements for robot exploration [10] where a leader is elected and the remaining robots follow the leader along some arbitrary path. But the objective function in such techniques is either at least once coverage or continual coverage of a terrain, both of which are not the goal of our thesis. Our work attempts to avoid revisits to cells during terrain coverage whenever possible.

3 Multi-robot Area Coverage Algorithms

A coverage area can be visualized as a grid consisting of $\mathbf{M} \times \mathbf{N}$ cells, each of square geometry and identical size. Representing an area in this form is called the occupancy grid representation. In this representation, a zero denotes free unexplored space and a non-zero value denotes either a covered cell or obstacles, as the case may be. Typically, positive numbers are used to represent robot visits and negative numbers are used to represent obstacles. A team of M homogeneous mobile robots, which can communicate with each other, is deployed for coverage. Each robot is identified with a globally unique ID. When the robots communicate, they exchange *'state'* information. We assume that the robots have the capability to sense the locations (or cell) and the boundaries when they reach them.

The presence of fewer entry points into the area requires that the team enter the region and spread out as quickly as possible to cover the grid while minimizing overlap. Overlap is defined as a visit by a robot to an already covered cell and by definition, it is cumulative. To measure this, we introduce a parameter called the overlap_ratio which is formally defined in eqn. (1). In order to minimize overlap, negotiation strategies must distribute work among the robots in a fair manner and ensure that robots independently cover as much area as possible before coordinating their actions to cover the remaining area. This requires robots to exchange and process a lot of information before they cover cell independently. It also necessitates robots to be predictable in their actions for which they must follow deterministic coverage patterns. As this is difficult to achieve and implement in simple communicating robots capable of scanning and maintaining cell status, we treat coordination as the exchange of a single message or a sequence of messages between the robots in the team, depending on their states.

$$overlap\_ratio = Number\ of\ cells\ in\ overlap \,/\, Total\ number\ of\ cells\ in\ grid \qquad (1)$$

Another challenge is that the robots often have knowledge only about the extent of this area and can sense/detect the adjacent cells. It forces the robots to take coverage decisions based on *their* knowledge of the environment. In such tasks, robots often do not have complete information about the area required to perform coverage optimally.

We have developed a class of multi-robot area coverage algorithms (on-line and off-line computations) to coordinate the robots for covering given areas minimizing repetitive visits to cells, measured using the overlap_ratio. Each algorithm is suited to specific types of robots and contexts. These are summarized in table 1.

Table 1. Area coverage algorithms and their requirements at a glance

Algorithm	Communication	Collision Avoidance	Minimum Distance	Computation
NCC	No	No	No	On-line
OSC	Yes	No	No	On-line
OSCARD	Yes	Yes	No	On-line
NJ	Yes	Yes	Yes	On-line
AAA	No	Yes	Yes	Off-line

In the NCC algorithm, each robot randomly decides its next step for movement. It is naïve, easy to implement and forms our basis for comparison with other multi-robot coverage algorithms. The next set of algorithms (except AAA) are all online algorithms based on the assumption that robots communicate at every step to synchronize their actions in order to minimize overlap. When a decision is taken, it is assessed by other team members thru inter-robot communication. If the action is acceptable to all robots in the team, then this action is performed by that robot. *Acceptable* refers to the act of ensuring that two or more robots are not present in the same cell at the same time. If such a situation occurs, a collision is deemed to have taken place.

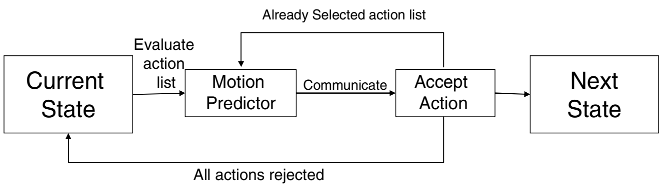

Fig. 2. Coordinated Robot Movement Strategy

The OSC algorithm employs one-step communication with other robots to avoid collisions. OSCARD employs one-step communication and assumes the capability for each robot to recognize already covered cells. Since, all robots are homogeneous, they cover a cell in identical manner and this information is exploited in this algorithm. The NJ algorithm maintains a fixed inter-robot distance in addition to having all capabilities and requirements of OSCARD. The threshold for inter-robot distance is determined by the grid size and number of robots in the team. Figure 2 summarizes the robot movement strategy of these online coordination algorithms.

Each algorithm is a different implementation of the "motion predictor module" which helps robots to avoid visiting already covered cells. Despite avoiding visited cells, a robot might get trapped and the algorithms have ways of breaking the deadlock at the cost of increased overlap. The AAA is an offline algorithm that assumes global grid knowledge and computes each robot's initial position in the grid using an equally-likely distribution. The robots then move in a deterministic manner to those positions and perform coverage along the axis of the grid. Their orientation is known a priori and the robots maintain the same throughout coverage.

4 Homogeneous Hierarchical Composition of Areas

Typical area coverage scenarios require a team of robots to cover very large areas. In such scenarios minimizing global overlap with distributed coordination mechanisms can be very difficult. One solution to this problem would be to try and localize the coordination required by adopting a divide and conquer approach. We accomplish that by dividing our robots into teams and spread the teams out to cover smaller units of the grid in parallel. In the context of multi-robot area coverage, the coverage grid can be split into smaller grids of fixed sizes where overlap ratio is controlled and repeat that pattern until coverage is complete in the entire grid. Each team will have a leader who is responsible to coordinating his team's actions with other teams in order to minimize overlap. This leader can be elected using any known leader election [12,15] methods. Multiple teams could, therefore, work independently while coordinating among themselves while performing coverage. This has a significant impact on saving communication cost between the robots during coordination. In our work on area coverage, we have devised a methodology to scale the algorithms to cover very large grids using algorithms described in section 3.

In hierarchies, we define the smallest grid used to decompose the area as a *primitive* which is covered by a team of robots. A *team* is defined as a group of robots using the same algorithm covering the same grid. The coverage of each primitive is achieved by the team synchronously. Grids formed as a consequence of stacking the primitives together give rise to larger grids which are called *grid-cells* at the higher levels in the hierarchy. A grid-cell can also be considered as a primitive grid at these higher levels covered by a larger team of robots. Hence, the rules that apply to a primitive apply at all the intermediate levels in the hierarchy. This is illustrated in Fig. 3.

Several such teams of robots occupy and cover the primitives in an order directed by some meta-level area coverage algorithm and in effect cover the entire area. In the level immediately above the primitives in the hierarchy, each primitive can also be treated as a cell and the each team of robots covering the primitive may be *treated* as a single "more powerful" robot covering that "cell". We may then use the same set of algorithms to cover the area at the next higher level in hierarchy. The coverage algorithms used *within a team* and *across teams* are totally decoupled and the best combination may be employed to minimize overlap. It is then sufficient to compute the optimal decomposition of a given coverage grid in terms of these small grids and obtain their initial positioning. We now present the crux of our work and introduce the hierarchical composition theorem (H^2C theorem) for theoretically computing the overlap ratio in very large grids using the empirical results obtained from overlap ratio for small grids of different configurations.

Theorem 1. *The overlap_ratio in covering an $N \times N$ grid using some $n \times n$ as the coverage primitive for two levels in a hierarchical manner is given by sum of the overlap ratios in the grids at the two levels.*

Theorem 2. *The overlap ratio, as obtained from Homogeneous Hierarchical Composition Theorem, in covering a region using smaller coverage grids in a multi-level hierarchy is given by*

$$O(m): x_0 + x_1 + x_2 + \ldots + x_m$$

where O(m) is the overlap ratio obtained at the m^{th} level in the hierarchy and x_0 through x_m are the overlap ratios obtained at the corresponding levels using the primitive grids

Note: Formal proof to the theorems is given in Appendix 1.

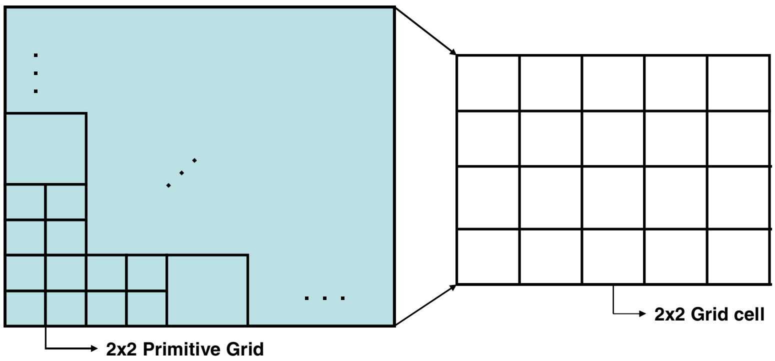

Fig. 3. Illustration of Hierarchical Area Composition and Meta-level view

4.1 Implication of Homogeneous Hierarchical Composition

An important consequence of the theorems stated above is that a grid may be decomposed into as many levels in hierarchy as required and in any order so as to minimize overlap. However, the number of levels in hierarchy will impact the number of robots needed and in turn the inter-robot communication cost to maintain synchrony. Another consequence is that, any coordination algorithm that satisfies the requirements in table 1 and performs cohesive team movement from one primitive to another can be used at any level(s) in coverage.

The theoretical and experimental results for overlap_ratio obtained through hierarchical decomposition deviate from one another as the experimental overlap had to take the team relocation into account while shifting from one sub-grid to another. These robot movements were assumed to be deterministic across the primitives and require communication between the various primitive team leaders to coordinate their respective team movements and minimize overlap. It can be shown that this deviation is bounded by a factor $m (n-1)/2n^2$, where m is size of small robot team and n is the size of primitive grid. For values of m and n chosen such that $m \leq n$, this factor is always less than 0.5. Therefore, the hierarchical composition framework effectively spreads the robots into teams across the grid and achieves coverage at acceptable levels of overlap ratio. Our framework was validated on large grids of size 1024×1024 for various team sizes and the results clearly indicated that the system can save over 90% of effort (computed as number of robot movements) even using the naïve NCC algorithm in covering the area.

4.2 Limitations of Homogeneous Hierarchical Composition

According to the theorems, we assume that the team of robots operating within one primitive grid function as a single unit at the immediate next higher level. This implies that while moving from one primitive to the next, all the robots must move as a cohesive unit from the current primitive to the same next primitive grid and in a deterministic manner. This requires the robots to maintain close coordination between them (even for naïve NCC coverage algorithm!). One should note that when the robot-collision avoidance is performed at an intermediate level in the hierarchy, it corresponds to a team of robots avoiding another team while shifting across primitives. This requires the leaders to communicate between themselves and maintain the required distance between their corresponding teams as dictated by the meta-level coverage algorithm. If the meta-level algorithm does not mandate communication, then leaders will perform basic communication to ensure that robot collisions and overlap across teams are minimized. While this operation amounts to overhead for using the hierarchical framework in area coverage, the alternative (direct coverage of the large grid) requires all robots to communicate with each other until they moved into mutually undisturbed positions. The overhead involved in achieving the latter far exceeds the cost of coverage in terms of number of steps to complete coverage or resources required. In comparison, the complexity would reduce by several orders of magnitude by using the leader election technique for inter-team coordination. It is always possible to restrict the number of such teams sent in to cover a given area and effectively reduce the communication overhead.

5 Homogeneous Hierarchical Composition Applied to large areas

The power of the H^2C theorem is highlighted when we study its performance in comparison to direct coverage of large grids using the same algorithms. In each of the following figures shown, the algorithm indicated was used in performing direct coverage as well as in hierarchical coverage. In the case of hierarchical coverage with several levels of hierarchy, the same algorithm was employed at all the levels with equal number of robots at the lower and each higher level.

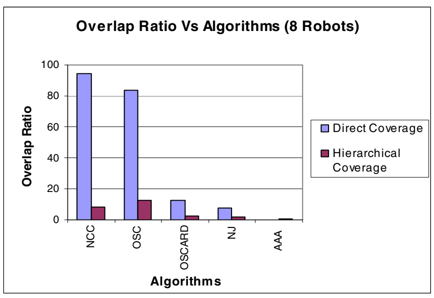

Fig. 4. Direct Vs Hierarchical Coverage of 64×64 Grid (8 Robots)

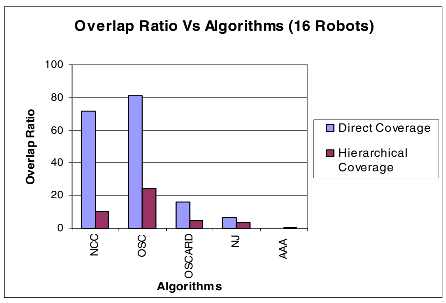

Fig. 5. Direct Vs Hierarchical Coverage of 64×64 Grid 16 Robots

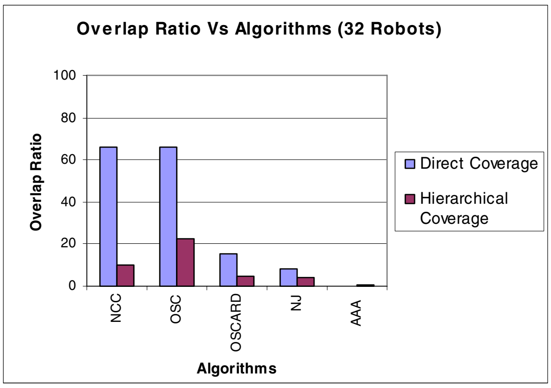

Fig. 6. Direct Vs Hierarchical Coverage of 64×64 Grid 32 Robots

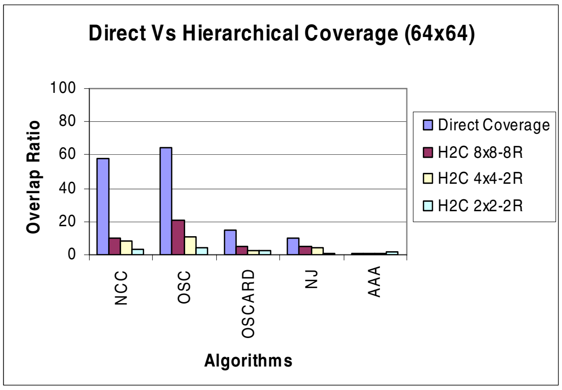

Fig. 7. Direct Vs Hierarchical Coverage of 64×64 Grid

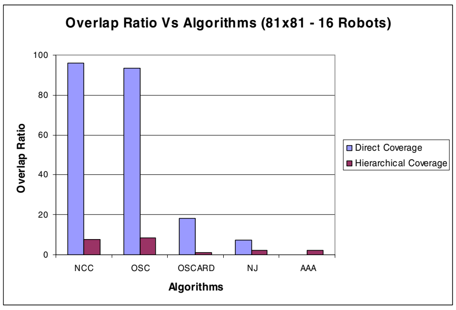

Fig. 8. Direct Vs Hierarchical Coverage of 81×81 Grid 16 Robots

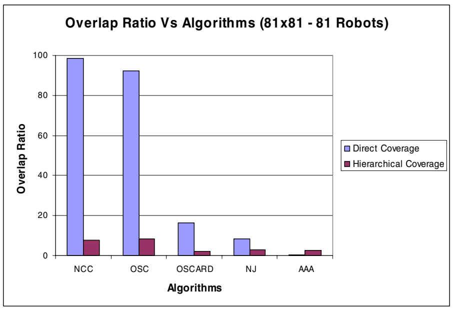

Fig. 9. Direct Vs Hierarchical Coverage of 81×81 Grid Using 81 Robots

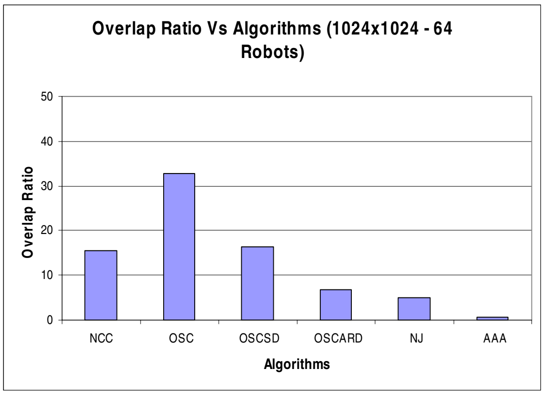

Fig. 10. Performance of Hierarchical Coverage of 1024×1024 Grid Using 64 Robots

5.1 Design of Experiments

It is noteworthy to mention that when the coverage related simulation experiments were conducted on grids of very large sizes (256×256, 512×512 and 1024×1024, for example), all algorithms other than the NCC algorithm did not run to completion even when simulated for over 36 hours. We inferred that this behavior was because all online algorithms barring the NCC algorithm require consensus through communication from all robots in the team at every step. This was significant communication overhead on the system and the robots were busy most of the time communicating to obtain acceptance. As a consequence, coverage rate was drastically slow and completion was never reached.

On the other hand, when the number of robots was decreased to an acceptable number, there are lesser number to perform coverage, most of which were attempting to cover the same section of the grid and unable to get out of this section. It must be

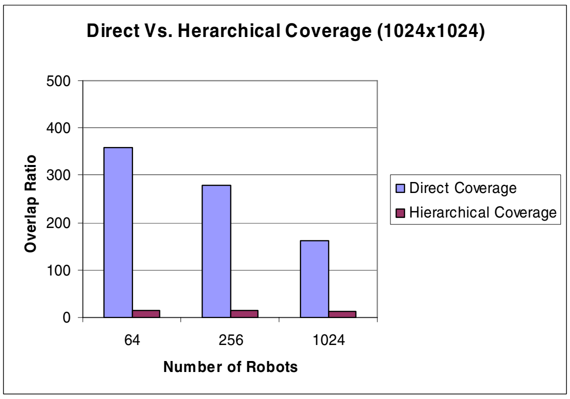

Fig. 11. Comparing Performance of Coverage for varying robot team sizes in 1024×1024 grid

noted that though the coverage algorithms (as compared with NCC) attempt to minimize visits to already covered cells, they still select randomly when there is more than one possibility. A comparison of the performance of direct coverage against hierarchical coverage is shown in Fig. 11 for 1024×1024 grid for varying number of robots using the NCC algorithm. As mentioned before, NCC algorithm was applied at all levels in the hierarchical case.

6 Summary and Conclusion

In this paper, we have discussed one approach to area coverage using multiple mobile robots. In our work, we developed a methodology to solve this problem in a coordinated manner using these robots. We briefly discussed the different coordination algorithms we designed with small robot teams and successively refined them to minimize overlap_ratio.

We then proposed a hierarchical framework for integrating these solutions for coverage of small areas to cover arbitrarily large areas and stated the Homogeneous Hierarchical Composition Theorem that states that the overlap ratio in coverage of a given area consisting of a grid of multiple primitive grids is the sum of the overlap ratios in coverage at the two levels. This result was further shown to be scalable to any number of levels in hierarchy.

The design of the experiments was explained and the results were presented. The performance of these algorithms was studied for varying number of robots in a team and for varying grid sizes and the various observations were listed. Results clearly indicate that this framework is very effective and for large grids, can save effort (measured by number of robot actions) even using simple coverage algorithms.

In future we plan to extend the results discussed in this paper to arbitrary sized areas and varying decompositions. Work is currently underway to extend the coordination algorithms to heterogeneous robots under lossy communication channels.

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Appendix 1: Homogeneous Hierarchical Composition Theorem

Proof for Theorem 1: *Consider a square grid of area $N \times N$ being covered by a team of M robots; let us assume that the grid can be composed using primitives of area $n \times n$. Then the $N \times N$ grid consists of $k^2 = (N \times N)/(n \times n)$ primitives (say). We position these k^2 grids in the form of a square to cover the entire grid. Let $M = a \times b$ and let each $n \times n$ grid be covered by 'a' robots. Then there are 'b' such teams to cover the $N \times N$ grid. Let these b teams be deployed in the k^2 grid for coverage. The problem is illustrated in Fig 12.*

Note: Although the proof refers to square grids for ease of exposition, these results apply equally well to rectangular shaped grids.

Let us suppose that the coverage of k^2 grid using a team of 'b' robots results in an overlap ratio of x_1. We obtain an overlap ratio of x_0 in the coverage of each primitive grid by a team of 'a' robots. Coverage of $k \times k$ grid necessitates the coverage of each primitive $n \times n$ and together, they guarantee the coverage of the $N \times N$ grid, as specified. Hence, the total number of cells in overlap is given by –

Number of cells in overlap for the primitive grid coverage $= (x_0 \times n^2) \times k^2$

Number of grid-cells in overlap during coverage of k × k grid = $(x_1 \times k^2)$

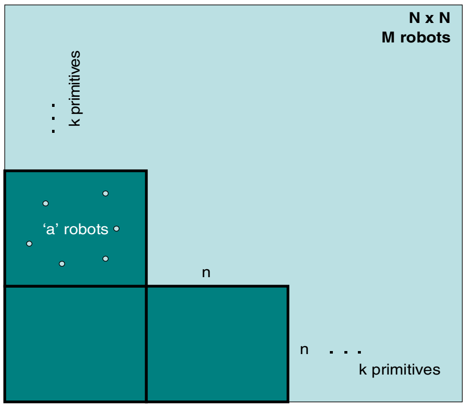

Fig. 12. Illustration of the H^2C theorem

Each cell of k × k grid is an n × n grid. Hence number of cells in overlap is given by – $= (x_1 \times k^2) \times n^2$

Total number of cells in overlap $= (x_0 + x_1) \times k^2 \times n^2$

Overlap ratio (By Definition) = (Total number of cells in overlap / Total number of cells)

$= (x_0 + x_1) \times k^2 \times n^2 / N^2$

$= (x_0 + x_1) \times N^2 / N^2$

$= (x_0 + x_1)$

This concludes the proof for Homogeneous Hierarchical Composition Theorem for obtaining the overlap in a large grid hierarchical composition of primitive grids. □

Proof for Theorem 2: *Proof by Principle of Mathematical Induction (PMI)*
P(1): The theorem holds true for single level in hierarchy.
O(1): $= x_0 + x_1$

Proof: Proved in Theorem 1. Therefore P(1) is true.
At the induction step, assume that the statement is true for some natural number m.
P(m): Theorem holds true for m levels in the coverage hierarchy. Therefore O(m): $x_0 + x_1 + x_2 + \ldots + x_m$ is true.

To show that P(m+1) is true whenever P(m) is true. Then by PMI, statement P(N) is true for all natural numbers N.

P(m+1): The theorem holds for m+1 levels in coverage hierarchy whenever P(m) is true.

O(m+1): $x_0 + x_1 + x_2 + \ldots + x_m + x_{(m+1)}$

From P(m), it is evident that any coverage grid can be composed for m levels in hierarchy and their overlap ratio can be obtained as the sum of overlap ratio at the corresponding levels. Let us now construct a grid of r^2 cells, each of which is an 'm' level coverage grid in hierarchy with a total of L cells. Let the actual number of cells in its side be M. Then we have the relation,

$$M^2 = r^2 \times L^2$$

Overlap ratio obtained in the L x L grid is given by –

$$O_L = [x_0 + x_1 + x_2 + \ldots + x_m]$$

Let the overlap at the highest level in hierarchy (m+1) be x(m+1). The total number of cells in overlap is then given by,

$$\begin{aligned} \text{Overall Overlap} &= [(x_0 + x_1 + x_2 + \ldots + x_m) \times L^2]\times r^2 + [x_{(m+1)}\times r^2]\times L^2 \\ &= [x_0 + x_1 + x_2 + \ldots + x_m + x_{(m+1)}] \times r^2\times L^2 \\ &= [(x_0 + x_1 + x_2 + \ldots + x_m + x_{(m+1)}]\times r^2\times L^2/M^2 \end{aligned}$$

Substituting from equation (1), we obtain,

$$\text{Overlap ratio} = [x_0 + x_1 + x_2 + \ldots x_m + x_{(m+1)}],$$

which proves our P(m+1) statement.

Therefore P(m+1) is true whenever P(m) is true. Hence, by PMI, statement P(N) is true for all Natural Numbers. This concludes the proof of Generalized H^2C theorem. □