On Directed Tree Realizations of Degree Sets

Prasun Kumar[1], Jayalal Sarma M.N.[1], and Saurabh Sawlani[2]

[1] Department of Computer Science and Engineering, IIT Madras, Chennai, India.
[2] Department of Electrical Engineering, IIT Madras, Chennai, India.

Abstract. Given a degree set $D = \{a_1 < a_2 < \ldots < a_n\}$ of non-negative integers, the minimum number of vertices in any tree realizing the set D is known[10]. In this paper, we study the number of vertices and multiplicity of distinct degrees as parameters of tree realizations of degree sets. We explore this in the context of both directed and undirected trees and asymmetric directed graphs (graphs which do not have a cycle of length two). We show the following results.

- We show a tight lower bound on the maximum multiplicity needed for any tree realization of a degree set.
- For the directed trees, we study two natural notions of realizability by directed graphs and show tight lower bounds on the number of vertices needed to realize any degree set.
- For asymmetric graphs, if $\mu_A(D)$ denotes the minimum number of vertices needed to realize any degree set, we show that $a_1 + a_n + 1 \leq \mu_A(D) \leq a_{n-1} + a_n + 1$. We also derive sufficiency conditions on a_i's under which the lower bound is achieved.
- We study the following algorithmic questions related to our problem and study their complexity. (1) Given a degree set D and a non-negative integer r (as 1^r), test whether the set D can be realized by a tree of exactly $\mu_T(D) + r$ number of vertices. We show that the problem is fixed parameter tractable under two natural parameterizations of $|D|$ and r. We also study the variant of the problem : (2) Given a tree T, and a non-negative integer r (in unary), test whether there exists another tree T' such that T' has exactly r more vertices than T and has the same degree set as T. We show that this problem can be solved in log-space.
- For directed trees, under the both notions of realizability, we show that if $\mu'(D)$ is the minimum number of vertices needed for any directed tree realization, then for any non-negative integer r, there is a directed tree with $\mu'(D) + r$ vertices realizing the same degree set.

1 Introduction

Representation of graphs is an important theme in various algorithmic design fronts for graph theoretic problems. The standard methods used are adjacency matrix and adjacency list representations. Since many applications require more succinct representation, degree sets and degree sequences have been considered where the uniqueness of the graph being represented can be traded off for succinctness. However, there is a host of computational [1] and combinatorial problems [10, 8, 9, 3] associated with these representations themselves.

In this context, we study tree realizations[3] of degree sets $D = \{a_1 < a_2 < \ldots < a_n\}$. It is known[10] that the minimum number of vertices necessary and sufficient for a graph to realize any degree set D is exactly $a_n + 1$. If the graph is restricted to be a tree, this is known [10] to be exactly $(\sum_{i=1}^{n}(a_i - 1)) + 2$ if and only if $a_1 = 1$.

We study the tree-realizability of degree sets under multiplicity constraints on each degree. That is, realizations where the multiplicity of each vertex is upper bounded by a number m. The realization that achieves the minimum number of vertices has exactly one vertex of each degree except for the degree 1. Hence the degree distribution is skewed. Can the degree set be realized by a tree with smaller maximum multiplicity if we are allowed to use more vertices? We answer this in the negative by arguing that the standard construction is indeed optimal in terms of maximum multiplicity.

Theorem 1. *The minimum multiplicity of* pendant vertices *in any tree realization for the degree set* $D = \{1 = a_1 < a_2 < \ldots < a_n\}$ *is* $\sum_{i=1}^{n} a_i - 2n + 3$.

We define the notion of degree multiset where each element repeats atleast m times. We also generalize the above theorem to the case of degree multiset.

We turn to degree set realizations using directed graphs. We study some natural variants. In the first variant, the degree set is said to be realized if there is a directed graph such that every vertex has either the in-degree *or* the out-degree from the set, and every number in the degree set appears as the in-degree of some vertex *or* as the out-degree of some vertex in the graph. We call this the $\vee$-REALIZATION of D. We observe a connection between this variant and the undirected graph realizations of degree sets in the case of bipartite graphs (in particular trees) and hence derive the minimum multiplicities in this case.

In the second variant, which we call the $\wedge$-REALIZATION of D, the degree set is to be realized by a directed graph such that every vertex has the in-degree *and* the out-degree from the set and every element in the set appears as the in-degree of some vertex *and* as the out-degree of some vertex. We prove the following theorem for directed tree $\wedge$-realizations.

Theorem 2. *The minimum order of any directed tree $\wedge$-realizing a degree set D is* $2(\sum_{i=2}^{n}(a_i - 1) + 1)$.

Relaxing the tree constraints, we study the $\wedge$ realizability of D in the context of asymmetric directed graphs[4]. These are classes of directed graphs where there are no cycles of length 2. We prove the following:

[3] See Section 2 for formal definitions.

[4] In [2], Chartrand *et al* studied directed asymmetric graph realizations of degree sets D. However, their definition of realization is only with respect to out-degree which differs from our definition

Theorem 3. *For any degree set $D = \{a_1 < a_2 < \ldots < a_n\}$, let $\mu_A(D)$ be the minimum number of vertices of any asymmetric directed graph realizing D, then:*

$$a_1 + a_n + 1 \le \mu_A(D) \le a_{n-1} + a_n + 1$$

We also give sufficient conditions among the a_i's under which the lower bound is achieved.

Turning to algorithmic questions : we consider the following Tree Extension Problem : Given a degree set D and a number r, test whether there is an undirected tree T that realizes the degree set D such that T has exactly $\mu_T(D)+r$ vertices. Here $\mu_T(D)$ denotes the minimum order of any tree realizing D.

From a known characterization of the realizability of D with a given number of vertices using the well-studied Frobenius problem, we show that the problem is polynomial-time many-one equivalent to Integer Knapsack Problem.

We study parametrized versions of the problem, with respect to two parameters - $|D|$, r. We show the following results.

Theorem 4. Tree Extension Problem *is fixed parameter tractable with respect to the parameters $|D|$ and r, when r is presented in unary.*

We study the following variant of the computational question. Given a tree T and a non-negative integer r, test whether there is another tree T' with the same degree set but now having exactly $|T| + r$ number of vertices. We show that this problem can be solved in log-space and hence in polynomial time.

The analogous problems for directed trees turn out to be surprisingly easier. We prove the following characterization. For a degree set D, let $\mu_\wedge(D)$ ($\mu_\vee(D)$) denote the minimum number of vertices required to $\wedge$-realize ($\vee$-realize) the degree set D using a directed tree.

Theorem 5. *Given a set D and a value r, the degree set can always be $\wedge$-realized (resp. $\vee$-realized) using a directed tree of $\mu_\wedge(D) + r$ (resp.$\mu_\vee(D) + r$ vertices.)*

2 Preliminaries

Let $G = (V, E)$ be a graph[5]. For $v \in V$, by $d(v)$ we denote the degree of the vertex v in G. A degree-set of a graph G (first studied by [10]) is a subset of $\mathbb{N}$[6] defined as follows: $D(G) = \{d(v) : v \in G\}$. A set $D \subset \mathbb{N}$ is said to be *realizable* if and only if there is a graph G such that $D(G) = D$.

The degree-sequence of the graph G is the sequence of numbers : $d(G) = (d(v) : v \in G)$. A sequence D with elements from $\mathbb{N}$ is said to be realizable if there is a graph whose degree sequence (up to the ordering) is $d(G)$. Several results are known about characterizing realizability of degree sequences using graphs and various subclasses of graphs[3, 9, 8].

[5] All graphs being considered in this paper are simple

[6] $\mathbb{N}$ includes 0.

Let $\mu(D)$ denote the minimum number of vertices that must be present in any realization of D. Let $\mu_T(D)$ denote the minimum number of vertices that must be present in any realization of D when the graph is restricted to be a tree.

In directed graphs, for a vertex v we denote by $d^-(v)$ and $d^+(v)$, the indegree and outdegree respectively. We write the indegree and outdegree for a vertex v_i as an ordered pair (a_i, b_i) which means $d^+(v_i) = a_i, d^-(v_i) = b_i$. A directed graph is said to be *asymmetric* if it does not have cycles of length two. Let $\mu_A(D)$ denote the minimum number of vertices that any graph realizing D must have. If it clear from the context, we drop the notation for type of realizability.

An intermediate case between degree sets and degree sequences is that of multiplicity-constrained degree sets, where we restrict the number of times that a vertex in the degree set appears in the realization. A natural restriction to study is when the multiplicity is bounded from above, given that the degree distribution in the realization of D with trees is highly skew. We also consider the complementary variant, where the multiplicity is bounded from below. In these cases, we can denote the degrees with a multi-set $D_m = \{a_1^{m_1}, a_2^{m_2}, \ldots, a_n^{m_n}\}$ where $a_i^{m_i}$ denotes that a_i is appearing at least m_i times in the multiset and m_is are positive integers. We now focus on a very special case of the degree multiset when $a_1 = 1, m_1 = 1$, which we need later in our construction. Under this assumption, $D_m = \{1, a_2^{m_2}, \ldots, a_n^{m_n}\}$. Since $1 \in D_m$, there exists a tree realization for D_m, we obtain the lower bound for any tree realizing D_m. We state (and prove in the appendix) the following proposition. The proof of this is an easy generalization of the argument in [10] which assumes each $m_i = 1$.

Proposition 1. *The minimum order of a tree realizing* $D_m = \{1, a_2^{m_2}, \ldots, a_n^{m_n}\}$ *is* $\sum_{i=2}^{n} m_i(a_i - 1) + 2$.

We briefly introduce the basics of parametrized complexity that we need in the paper. We refer the reader to a standard textbook[6] for details. A parametrized computational problem instance is denoted by (I, k) where k is the parameter. A problem is fixed parameter tractable (FPT) with respect to the parameter k if there is an algorithm solving the problem in time $f(k).n^{O(1)}$ where n is the size of the instance. In general, the parameter k is not unique. That is, it is possible to parametrize a problem in more than one way and using more than one parameter.

3 Multiplicity Lower Bounds in Tree-realizations

In this section, we prove lower bounds for the multiplicities of the pendant vertices (vertices of degree 1) in any realization of a degree set D using trees. We prove Theorem 1.

Theorem 6. *The minimum multiplicity of* pendant vertices *in any tree realization for the degree set* $D = \{1 = a_1 < a_2 < \ldots < a_n\}$ *is* $\sum_{i=1}^{n} a_i - 2n + 3$.

Proof. The set $D = \{1 = a_1 < a_2 < \ldots < a_n\}$ can be realized by a tree[10]. Minimum order of such a tree is $\sum_{i=1}^{n}(a_i - 1) + 2$. In minimum order tree construction, each a_i is connected with exactly $a_i - 2$ pendant vertices for $i = 3, 4, \ldots, n-1$ and for $i = 2$ *and* n, a_i's are connected with $a_i - 1$ pendant vertices and then a_i is connected with a_{i+1} for $i = 2, \ldots, n-1$.

Let m_i be the multiplicity of a_i in a tree realization T. Then, $(1^{m_1}, a_2^{m_2}, \ldots, a_n^{m_n})$ will be the degree sequence of T.

Case 1 when $a_2 \geq 3$. We recall that, if degree sequence $d = (d_1 \geq d_2 \geq \ldots \geq d_n)$ is being realized by a tree then number of pendant vertices in any tree realization [1] of d is $\sum_{i=1}^{k}(d_i - 2) + 2$ where k is the largest index such that $d_k \geq 3$. Hence, $m_1 = 2 + (a_2 - 2)m_2 + (a_3 - 2)m_3 + \ldots + (a_n - 2)m_n$, $\forall i$ $m_i \geq 1$. m_1 will be minimum if $m_i = 1$ for each $i = 2, 3, \ldots, n$ and the tree construction described above meets exactly this requirement. So minimum value $m_1 = 2 + (a_2 - 2) + (a_3 - 2) + \ldots + (a_n - 2) = \sum_{i=1}^{n} a_i - 2(n-1) + 1$ $= \sum_{i=1}^{n} a_i - 2n + 3$

Case 2 : when $a_2 = 2$. We first construct the tree for the degree set $D_1 = \{1 = a_1 < a_3 < \ldots < a_n\}$ in the way mentioned above and then introduce a vertex v. Now make v adjacent to any one pendant vertex,say u,so that v becomes the new pendant vertex and $d(u) = 2$. Degree set of this modified tree is D and number of pendant vertices is same as that in the tree realization of D_1 which is same as $m_1 = 2 + (a_3 - 2) + (a_4 - 2) + \ldots + (a_n - 2 = 2 + (a_2 - 2) + (a_3 - 2) + \ldots + (a_n - 2) = \sum_{i=1}^{n} a_i - 2(n-1) + 1 = \sum_{i=1}^{n} a_i - 2n + 3$

The above lemma can be generalized to the case of multisets. We give the proof in the appendix.

Theorem 7. *The minimum multiplicity of* pendant vertices *in any tree realization for the degree multiset* $D_m = \{1, a_2^{m_2}, \ldots, a_n^{m_n}\}$ *is* $\sum_{i=2}^{n} m_i(a_i - 2) + 2$.

4 Minimum-order Realizability of Directed Trees

In this section we explicitly compute the minimum number of vertices needed to $\wedge$-realize (resp. $\vee$-realize) the given degree set D using directed trees.

We describe $\vee$-realizability first. We prove the following general upper bound for $\mu_\vee(D)$. Let $\mu_B(D)$ denote the minimum number of vertices for any undirected bipartite graph realizing the degree sequence D. Given any undirected bipartite realization of a degree set by a graph $G = (U, V, E)$ we assign directions from U to V. This gives a $\vee$-realization of the same graph using a directed bipartite graph. Thus, we have the following proposition.

Proposition 2. $\mu_\vee(D) \leq \mu_B(D)$

Indeed, this proposition holds for directed trees as well since the underlying undirected graph is bipartite. We now argue that this upper bound is tight for trees and show the following theorem.

Theorem 8. *For the degree set $D = \{1 = a_1 < a_2 \ldots < a_n\}$, minimum order of a directed tree $T(V,E)$ so that $\forall v \in V, d^+(v) \in D$ or $d^-(v) \in D$, and for each $a_i \in D$ there is a vertex $u \in V$ such that $d^+(u) = a_i$ or $d^-(u) = a_i$, is same as the minimum order undirected tree realizing D,i.e. $\sum_{i=1}^{n}(a_i - 1) + 2$.*

Proof. The upper bound follows from the above proposition through the undirected tree-realization of D with optimal number of vertices.

Now we need to prove a lower bound on the order of a directed tree satisfying the given constraints and then give a realization which meets this bound.

For each i, $a_i \in D$ will appear as both (a_i, a_j) and (a_k, a_i) at least once, where $a_j, a_k \in D$. Thus, $1 \leq a_i + a_j \leq 2a_n$. Let $T(V,E)$ be a directed tree for D satisfying the constraints. We have,

$$\sum_{v \in V}(d^-(v) + d^+(v)) = 2|E| = 2(V-1) \geq \sum_{i=1}^{n} a_i + (V - n)$$

This implies the lower bound $|V| \geq 2 + \sum_{i=1}^{n}(a_i - 1)$.

Now we turn to $\wedge$-realizability of D using directed trees. It can be noted that a necessary condition is $0 \in D$ since the tree has to contain leaf nodes whose in-degree or out-degree has to be 0.

Theorem 9. *For the degree set $D = \{0 < 1 < a_2 < \ldots < a_n\}$, the minimum order of a directed tree T which $\wedge$-realizes the degree set D, is $2\left(\sum_{i=1}^{n}(a_i - 1)\right) + 2$.*

Proof. We prove the upper bound by constructing the directed tree. Construct a path with $2(n-1)$ number of vertices, say $u_1, u_2, \ldots, u_{2n-2}$. Now add $(a_2 - 1)$ pendant vertices to u_1. For each $2 \leq i \leq 2n-1$, add $a_{\lceil \frac{i}{2} \rceil + 1} - 2$ pendant vertices to u_i. Add $a_n - 1$ pendant vertices to the u_{2n-2}.

In this tree, first 2 vertices are having degree a_2, next 2 vertices are having degree a_3 and so on. Now we assign directions. Start with the first vertex u_1 in the path. Direct all edges connected with u_1 towards u_1. For the next vertex in the path u_2 assign directions to all adjacent edges away from u_2. Repeat this process to assign direction to all edges. Since each a_i, for $i = 2, 3, \ldots, n$, appears exactly twice and because of the way we are assigning directions to edges, a_i once appears as $(a_i, 0)$ and once as $(0, a_i)$ in final directed tree. For pendant vertices in undirected graph, indegree and outdegree pair occurs as either $(1, 0)$ or $(0, 1)$.

To prove the minimality, we first observe that the number of vertices in the above construction is $|V| = \sum_{i=2}^{n} 2(a_i - 1) + 2$. Now, consider the corresponding degree multiset $\{1, a_2^2, a_3^2, \ldots, a_n^2\}$. Applying proposition 1 with $m_i = 2 \forall i$ gives a matching lower bound on $|V|$.

5 Minimum order $\wedge$-realizability of Asymmetric Graphs

In this section we study $\wedge$ realizations of degree sets with asymmetric directed graphs. We introduce a notation for convenience in this section. For a directed graph G, let $\mathcal{A}_G$ denote the set that is $\wedge$-realized by G. Since the realizability is fixed, we drop it from the notation. Recall that $\mu_A(D)$ denotes the minimum order of any asymmetric directed graph realizing D. We start with a simple case which is similar to the starting point in [2].

Lemma 1. *If $D = \{a\}$ where a is a non-negative integer, then $\mu_A(D) = 2a+1$.*

Proof. This case is similar to [2]. When $a = 0$ the graph is an isolated vertex. For $a \geq 1$, all vertices in a directed graph with $\mathcal{A}_G = \{a\}$ must have both indegree and outdegree equal to a. Consider a vertex v, since the graph is asymmetric, v is connected to $2a$ distinct vertices. Accounting for these vertices and v, we have $2a+1$ vertices. Hence, $\mu_A(D) \geq 2a+1$. To complete the proof, we need to prove that $\mu_A(D) \leq 2a+1$. To do this, we will come up with a construction of a directed graph with $\mathcal{A}_G = \{a\}$ and order $2a+1$.

We define G to be the directed graph with the vertex set $\{v_1, v_2, \ldots, v_{2a+1}\}$. The edges are as follows: $\{(v_i, v_j) | 1 \leq i \leq 2a+1$ and $i+1 \leq j \leq i+a\}$ (where subscripts are modulo $2a+1$). Clearly, G is asymmetric and has $2a+1$ vertices with $\mathcal{A}_G = \{a\}$. Hence the proof.

Theorem 10. *If $D = \{a_1 < a_2 < \ldots < a_n\}, n \geq 2$ is a set of positive integers then*

$$a_1 + a_n + 1 \leq \mu_A(D) \leq a_{n-1} + a_n + 1.$$

Proof. We know that there is at least one vertex v of G with either indegree or outdegree equal to a_n. Without loss of generality, let us assume that $d^+(v) = a_n$. Now, we know that $d^-(v) \geq a_1$. Therefore, $d^+(v) + d^-(v) \geq a_n + a_1$. Since G is also asymmetric, it implies that the order of G is at least $a_1 + a_n + 1$.

To prove that $\mu_A(S) \leq a_{n-1} + a_n + 1$, we proceed by induction. By Lemma 1, we know that $\mu_A(\{a_1\}) = 2a_1 + 1$. Let the graph representing this be G_1. Divide G_1 into three components, C_x, C_y - each containing a_1 vertices, and C_z - containing the remaining vertex. From G_1, we obtain G_2, by adding a new component C_1 containing $a_2 - a_1$ vertices and adding the following edge set $E = \{(v_x, v_1) | v_x \in C_x \wedge v_1 \in C_1\} \cup \{(v_1, v_y) | v_1 \in C_1 \wedge v_y \in C_y\}$. Thus, we have an asymmetric directed graph for the degree set $\{a_1 < a_2\}$ with order a_1+a_2+1.

Now consider that there exists an asymmetric directed graph G_{n_0} with $\mathcal{A}_G = \{a_1 < a_2 < \ldots < a_{n_0}\}$, with order $a_{n_0-1} + a_{n_0} + 1$. G_{n_0} contains a total of $2n_0$ components :

- C_{n_0-1}, containing $a_{n_0} - a_{n_0-1}$ vertices with outdegree and indegree equal to a_1.
- C_i, for i from 1 to $n_0 - 2$, each containing $a_{i+1} - a_i$ vertices with outdegree a_1 and indegree a_{n_0-1-i}.
- C'_j, for j from 1 to $n_0 - 2$, each containing $a_{j+1} - a_j$ vertices with outdegree a_{n_0-1-j} and indegree a_1.
- C_x, containing a_1 vertices with outdegree a_{n_0} and indegree a_{n_0-1}.
- C_y, containing a_1 vertices with outdegree a_{n_0-1} and indegree a_{n_0}.
- C_z, containing 1 vertex with outdegree and indegree a_1.

From G_{n_0}, we obtain G_{n_0+1}, by adding two new components - C_{n_0} containing $a_{n_0+1} - a_{n_0}$ vertices, and C'_{n_0-1} containing $a_{n_0} - a_{n_0-1}$ vertices, and adding the edge set $E = E_1 \cup E_2 \cup E_3$, where

- $E_1 = \{(v_x, v_{n_0})|v_x \in C_x \wedge v_{n_0} \in C_{n_0}\} \cup \{(v_{n_0}, v_y)|v_{n_0} \in C_{n_0} \wedge v_y \in C_y\}$
- $E_2 = \{(v_y, v_{n_0-1})|v_y \in C_y \wedge v_{n_0-1} \in C'_{n_0-1}\} \cup \{(v_{n_0-1}, v_x)|v_{n_0-1} \in C'_{n_0-1} \wedge v_x \in C_x\}$
- $E_3 = \{(v_i, v'_i)|v_i \in C_i \wedge v'_i \in C'_{n_0-1-i}\}$, where $i \in \{1, 2, \ldots, n_0 - 2\}$

We can observe that G_{n_0+1} resembles G_{n_0} if n_0 is replaced with n_0+1. Thus, through this construction, we have proved that there always exists a asymmetric directed graph G with $\mathcal{A}_G = (a_1 < a_2 < \ldots < a_n)$, of order $a_{n-1} + a_n + 1$. Hence, the minimum order $\mu_A(D) \leq a_{n-1} + a_n + 1$.

We now identify a condition that is sufficient in order to achieve the lower bound in theorem 10.

Lemma 2. *If $D = \{a_1 < a_2 < \ldots < a_n\}, n \geq 2$ is a set of positive integers which satisfies the following condition:*

$$a_i + a_{n+1-i} = a_j + a_{n+1-j} \forall i < j$$

then $\mu_A(D) = a_1 + a_n + 1$.

Proof. From Theorem 10, we know that $\mu_A(D) \geq a_1 + a_n + 1$. So, we only have to show that, if the given condition is satisfied, $\mu_A(D) \leq a_1 + a_n + 1$. To do this, we will come up with a construction of a directed graph G with $\mathcal{A}_G = \{a_1 < a_2 < \ldots < a_n\}$ and order $a_1 + a_n + 1$.

D satisfies the given condition. We shall construct a directed graph with order $a_1 + a_n + 1$ by induction on n.

For $n = 2$, $a_1 + a_n + 1 = a_{n-1} + a_n + 1$. Therefore, by Theorem 10, we can always construct a directed graph for $n = 2$ with order $a_1 + a_2 + 1$.

Now take $n = 3$, define G to be the directed graph with $V(G) = \{v_1, v_2, \ldots, v_{a_1+a_3+1}\}$ and $E(G) = \{(v_1, v_j)|2 \leq j \leq a_1 + 1\} \cup \{(v_j, v_1)|a_1 + 2 \leq j \leq a_1 + a_3 + 1\} \cup \{(v_{a_1+a_3+1}, v_j)|2 \leq j \leq a_3\} \cup \{(v_j, v_{a_1+a_3+1})|a_3 + 1 \leq j \leq a_1 + a_3\}$. G has $a_1 + a_3 - 1$ vertices of indegree and outdegree 1. Since we know that the given condition is satisfied, $a_1 + a_3 = 2a_2$ and $a_1 + a_3 - 1 = 2(a_2 - 1) + 1$. From Lemma 1 1, we can construct a directed graph G_1 of order $a_1 + a_3 - 1$ with $\mathcal{A}_{G_1} = \{a_2 - 1\}$. Superimposing G_1 on the vertices with outdegree and indegree 1 in G, we get a directed graph for $n = 3$ with order $a_1 + a_3 + 1$.

Now, let us assume that such a construction is possible for $n = m$. We will try to construct a graph of order $a_1 + a_n + 1$ for $n = m + 2$. Define G to be the directed graph with $V(G) = \{v_1, v_2, \ldots, v_{a_1+a_n+1}\}$ and $E(G) = \{(v_1, v_j)|2 \leq j \leq a_1 + 1\} \cup \{(v_j, v_1)|a_1 + 2 \leq j \leq a_1 + a_n + 1\} \cup \{(v_{a_1+a_n+1}, v_j)|2 \leq j \leq a_n\} \cup \{(v_j, v_{a_1+a_n+1})|a_n + 1 \leq j \leq a_1 + a_n\}$. G has $a_1 + a_n - 1$ vertices of indegree and outdegree 1. Since we know that the required condition is satisfied, $a_1 + a_n = a_2 + a_{n-1}$ and $a_1 + a_n - 1 = (a_2 - 1) + (a_{n-1} - 1) + 1$.

From our induction assumption, we can construct a graph G_1 of order $a_1 + a_n - 1$ with $\mathcal{A}_{G_1} = \{a_2 - 1, a_3 - 1, \ldots, a_{n-1} - 1\}$ (because G_1 has m number of vertices. Superimposing G_1 on the vertices with outdegree and indegree 1 in G, we get the desired graph for $n = m + 2$. This completes the construction and the proof.

We are able to prove exact bounds for a special case of the degree set.

Lemma 3. *If $D = \{0, a_2\}$, then $\mu_A(D) = 2a_2$.*

Proof. Consider a directed asymmetric graph G for which $\mathcal{A}_G = \{0, a_2\}$. We know that G has at least one vertex, say v_1, with outdegree equal to a_2. Its indegree can be equal to either 0 or a_2. Consider the case in which its indegree is a_2. Since the graph is asymmetric, v_1 connects to $2a_2$ distinct points. Thus the order of G in this case would be at least $2a_2 + 1$. Now, consider the case where $d^-(v_1) = 0$. Here, v_1 connects to a_2 vertices (say $v_2, v_3, \ldots, v_{a_2+1}$), whose indegrees now cannot be equal to 0, and so are all equal to a_2. So, v_2 has edges coming in from $a_2 - 1$ vertices apart from v_1. If any of these vertices are one of $v_2, v_3, \ldots, v_{a_2+1}$, then that particular vertex would have both indegree and outdegree equal to a_2, realizing our earlier case and thus making the order of G at least $2a_2 + 1$. However, if v_2 does not connect to any of $v_2, v_3, \ldots, v_{a_2+1}$, then it connects to $a_2 - 1$ new vertices $(v_{a_2+2}, v_{a_2+3}, \ldots, v_{2a_2})$. Thus the order of G would be at least $2a_2$. From the above cases, we can see that the order of the directed graph must be at least $2a_2$, i.e. $\mu_A(\{0, a_2\}) \geq 2a_2$. To complete the proof, we need to prove that $\mu_A(\{0, a_2\}) \leq 2a_2$. To do this, we will come up with a construction of a directed graph with $\mathcal{A}_G = \{0, a_2\}$ and order $2a_2$.

Define G to be the directed graph with $V(G) = \{v_1, v_2, \ldots, v_{2a_2}\}$ and $E(G) = \{(v_i, v_j) | 1 \leq i \leq a_2 \text{ and } a_2 + 1 \leq j \leq 2a_2\}$. Then G is asymmetric with order $2a_2$ and $\mathcal{A}_G = \{0, a_2\}$. Hence, the proof.

6 Complexity results on Tree Extension Problem

We argue complexity results on the following algorithmic problems related to degree set realizations of trees. We define the problems formally first.
Tree Extension Problem(TEP) : Given a degree set D and an integer r, test if there is a tree having $\mu_T(D) + r$ vertices that realizes the degree set D.
Unary Tree Extension Problem (UTEP) : Given a tree T on ℓ vertices and a string 1^r, test if there is another tree T' having exactly $\ell + r$ vertices and the degree set same as that of T.

One important ingredient of our arguments about complexity of the above stated problems is the following combinatorial connection first proved by Gupta *et al*[7] between realizability and the well-studied Frobenius problem. We state it differently here, but the proof can be derived from the proof of Theorem 3 in [7]. However, we also give an alternative proof for the forward direction.

Lemma 4 ([7]). *If the degree set $D = \{a_1 = 1 < a_2 < \ldots < a_n\}$ is realized by a tree $T(V, E)$ then we can get another tree realization $T_1 = (V_1, E_1)$ where*

$|V_1| = |V| + r$,*r is a positive integer,* if and only if r *is a linear combination of* $(a_i - 1)$, *i.e.*

$$r = \sum_{i=2}^{n} k_i(a_i - 1) \tag{1}$$

where k_i's are non-negative integers.

Proof. Without loss of generality, we fix $T(V, E)$ as the tree with minimum order realizing D. Let m_i be the multiplicity of vertices with degree a_i in T. Hence, $m_i = 1$, for each $2 \leq i \leq n$ and $m_1 = \sum_{i=1}^{n} a_i - 2n + 3$ which is also the minimum multiplicity of pendant vertices in any tree realization. Now we add r number of vertices so that exactly k_i vertices are produced with degree a_i, where k_i's are non-negative integers, to get $T_1 = (V_1, E_1)$ and hence $r = \sum_{i=1}^{n} k_i$. So $\{1^{\sum_{i=1}^{n} a_i - 2n+3+k_1}, a_2^{1+k_2}, \ldots, a_i^{1+k_i}, \ldots, a_n^{1+k_n}\}$ is the degree sequence of T_1 . Consider two following cases:

Case 1 : $a_2 \geq 3$. By the bounds from [1], $\sum_{i=1}^{n} a_i - 2n + 3 + k_1 = \sum_{i=2}^{n}(k_i + 1)(a_i - 2)$ From this we get $k_1 = \sum_{i=2}^{n} k_i(a_i - 2)$. Hence $r = \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} k_i(a_i - 1)$.

Case 2 : $a_2 = 2$. Since $(a_2 - 2) = 0$ so $\sum_{i=3}^{n}(k_i+1)(a_i-2) = \sum_{i=2}^{n}(k_i+1)(a_i-2)$. Hence we will get the same result.This completes the proof.

Using the above Lemma, we show the following theorem:

Theorem 11. Unary Tree Extension Problem *can be solved in log-space.*

Proof. We prove the theorem by reducing the problem to unary subset sum problem which can be solved in log-space. The unary subset sum problem is defined as follows. Given a (multi)-set S of m integers $b_1, b_2, \ldots b_m$ and a value c (all inputs in unary) test if there is a subset S' of these integers such that $\sum_{i \in S'} b_i = c$. The reduction runs in log-space as follows. For $1 < i \leq n$, let $t_i = \lceil \frac{r}{a_i - 1} \rceil$. Given a tree T and r in unary, write down the following set S and r in unary, choose $c = r$ and define:

$$S = \bigcup_{i=2, j=1}^{i=n, j=t_i} \{(a_i - 1)j\}$$

Indeed, if there is a subset of S that sums up to r, then it is clear that this choice of the j's satisfies equation 1. Any solution for the k_i's in equation 1, it must be that $k_i \leq t_i$ for all i. Hence the corresponding terms $k_i(a_i - 1)$ will appear in the set S as well. Choosing these terms in S' ensures $\sum_{i \in S'} b_i = r = c$. To argue the complexity of the reduction, notice that we can compute a_i's each time on the fly by enumerating the degree up to the maximum degree. This can be done in log-space.

The idea in the above proof can be adapted to argue that TREE EXTENSION PROBLEM is equivalent to INTEGER KNAPSACK PROBLEM(IKP) which can be stated as follows : Given non-negative integers $c_1, \ldots, c_k$, and a value d - the problem asks if there are non-negative integers $d_1, d_2, \ldots, d_k$ such that $\sum_i c_i d_i = d$. Given a degree set D, consider the IKP instance with $k = |D| - 1$ and $c_i = a_{i+1} - 1$ for all $1 \leq i \leq k$. Choose $d = r$. In the reverse direction, given non-negative integers $c_1, \ldots, c_k$, and a value d, consider the degree set $D = \{1, c_1, \ldots, c_k\}$ and $r = d$. The correctness of the reductions follow from Lemma 4 directly. This discussion gives us the following proposition.

Proposition 3. TREE EXTENSION PROBLEM *is equivalent to* INTEGER KNAPSACK PROBLEM.

We consider two natural parameterizations of TREE EXTENSION PROBLEM and argue theorem 4.
Parameterizing with respect to $|D|$ when r is given in unary : In this setting, we give a reduction to VARIETY SUBSET SUM PROBLEM. The variety subset-sum problem : given a multiset A, and a target sum b, the problem asks if there is a sub(multi)set of A that adds up to exactly b. To do the reduction, we will list down the number $(a_i - 1)$ where $a_i \in A$, exactly r number of times in the subset. Since r is given in unary we can, in polynomial time, write out these numbers. There will be exactly nr of them. The correctness and resource bounds of the reduction follow easily.
The VARIETY SUBSET SUM PROBLEM was shown[4] to be fixed-parameter tractable with respect to the number of distinct elements in A as the parameter. As we can see in the above case, this is precisely $|D| - 1$. Hence TREE REALIZABILITY PROBLEM is fixed-parameter tractable with respect to $|D|$ as the parameter.
Parameterizing with respect to r as the parameter, when r is given in unary We first notice that VARIETY SUBSET SUM PROBLEM reduces to MAXIMUM KNAPSACK PROBLEM. We define the problem first. Given a set $\{x_1, x_2, \ldots x_n\}$ with sizes $s_1, s_2, \ldots s_n$ and profits $p_1, p_2, \ldots p_n$ respectively, and two values knapsack capacity b and profit threshold k - test if there exists a subset $S \subseteq [n]$ such that : $\Sigma_{i \in S} s_i \leq b$ and $\Sigma_{i \in S} p_i \geq b$. To reduce VARIETY SUBSET SUM PROBLEM, given $A = \{a_1, a_2, \ldots a_n\}$ and target sum t, produce x_i's such that $s_i = p_i = a_i$ and $b = p = t$. The inequalities ensures that the Maximum Knapsack Problem has a solution if and only if there is a subset $A' \subseteq A$ which adds up to exactly t. Fernau[5] has shown that the MAXIMUM KNAPSACK PROBLEM is fixed parameter tractable with respect to the parameter b. Since our reduction maps the parameter t to exactly p, this shows that the TREE EXTENSION PROBLEM is fixed parameter tractable with respect to the parameter r when r is given in unary.

6.1 Tree Extension Problem for Directed Trees

In this section we address similar computational problem for directed trees under the $\wedge$-realizability and the $\vee$-realizability. Surprisingly in both cases, it turns out to be the case that for every non-negative integer r, there are directed trees with

$\ell + r$ vertices $\wedge$-realizing and $\vee$-realizing (where ℓ takes appropriate values from Theorem 8 and Theorem 9 respectively). We prove these two results now.

Theorem 12. *For the degree set $D = \{0, 1, a_2, \ldots, a_n\}$ if we have a directed tree realization $T_d(V_d, E_d)$ then we can have another tree realization*[7] *$T_{d_1} = (V_{d_1}, E_{d_1})$ where $|V_{d_1}| = |V_d| + r$ for each non-negative integer r.*

Proof. Without loss of generality, we fix $T_d(V_d, E_d)$ as the directed tree with minimum order realizing D. We now consider two cases depending on the number of pendant vertices, say V_p, in T_d:

Case 1: when $r \leq V_p$
Add r number of pendant vertices to any r number of already existing pendant vertices in T_d so that if $d^-(p) = 1$, make p adjacent to newly added vertex by an outgoing edge and similarly if $d^+(p) = 1$, make p adjacent to newly added vertex by an incoming edge. Since $0, 1 \in D$, degree set remains unchanged and we get another tree T_{d_1} with k vertices more than T_d.

Case 2 : when $r \geq V_p$, let $r = l * V_p + r_0$ where l is a positive integer ≥ 1 and r_0 is another non-negative integer $\leq V_p - 1$.
First add V_p number of pendant vertices to T_d in the way described in case 1 and repeat the same procedure $(l-1)$ times more with directed tree obtained from the previous iteration and in the process degree set also does not change as explained above. In last iteration, we will do the same for remaining r vertices. This completes the proof.

References

1. R. Arikati, Srinavasa and Anil Maheshwari. Realizing degrees sequences in parallel. *SIAM Journal of Discrete Mathematics*, 9:317–338, 1996.
2. Gary Chartrand, Linda Lesniak, and John Roberts. Degree sets for digraphs. *Periodica Mathematica Hungarica*, 7:77–85, 1976.
3. P. Erdös and T. Gallai. Graphs wiyh prescribed degrees of vertices. *Mat. lapok*, 11:264–274, 1960.
4. Michael Fellows, Serge Gaspers, and Frances Rosamond. Parameterizing by the number of numbers. In Venkatesh Raman and Saket Saurabh, editors, *Parameterized and Exact Computation*, volume 6478 of *Lecture Notes in Computer Science*, pages 123–134. Springer Berlin / Heidelberg, 2010.
5. H. Fernau. Parameterized algorithms: A graph-theoretic approach. Technical report, Universitat Tubingen, Tubingen, Germany, 2005.
6. J. Flum and M. Grohe. *Parameterized Complexity Theory (Texts in Theoretical Computer Science. An EATCS Series)*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2006.
7. Gautam Gupta, Puneet Joshi, and Amitabha Tripathi. Graphic sequences of trees and a problem of frobenius. *Czechoslovak Mathematical Journal*, 57:49–52, 2007.
8. louis Hakimi, S. On realizability of a set of integers as degrees of the vertices of a linear graph i. *SIAM Journal of Discrete Mathematics*, 10:496–506, 1962.

[7] For $\vee$-realizability, we do not require 0 to be in D.

9. V. Havel. Eine bemerkung über die existenz der endlichen graphen. *ĉasopis pêst. Mat.*, 80:477–480, 1955.
10. S.F. Kapoor, A.D. Polimeni, and C.E. wall. Degree sets for graphs. *Fundamental Mathematics*, 95:189–194, 1977.

A Proof of Theorem 7

Proof. Let k_1 be the multiplicity of pendant vertices and $(m_i + k_i)$ be the multiplicities of remaining a_i^s in a tree realization T,where $\forall i \in [n] k_i$ is a non-negative integer, then $1^{k_1}, a_2^{m_2+k_2}, \ldots, a_n^{m_n+k_n}$ will be the degree sequence of T.

Case 1: when $a_2 \geq 3$.

$$k_1 = 2+(a_2-2)(m_2+k_2)+(a_3-2)(m_3+k_3)+\ldots+(a_n-2)(m_n+k_n), \forall i k_i \geq 0$$

Since each term in the right hand side is a positive integer and there must exist at least m_i vertex with degree a_i in T so k_1 will be minimum if $k_i = 0$ for each $i = 2, 3, \ldots, n$ and the tree construction mentioned in *Lemma 1* meets exactly this requirement. So, the minimum value of

$$\begin{aligned} k_1 &= 2 + (a_2 - 2)m_2 + (a_3 - 2)m_3 + \ldots + (a_n - 2)m_n \\ &= \sum_{i=2}^{n}(a_i - 2)m_i + 2 \end{aligned}$$

Case 2: when $a_2 = 2$ Since $(a_2 - 2) = 0$ so we will get the same value as in case 1 using a similar argument.

B Proof of Proposition 1

Proof. Assume $T(V, E)$ realizes D_m and $|V| = v$ so

$$\sum_{v \in V} d(v) = 2|E| = 2(v-1) \geq 1 + \sum_{i=2}^{n} m_i a_i + (v - \sum_{i=2}^{n} m_i - 1)$$

. From this we get $v \geq \sum_{i=2}^{n} m_i(a_i-1)+2$. We now give a procedure to construct a tree which exactly matches this bound. First we construct a path with $\sum_{i=2}^{n} m_i$ number of vertices, now add (a_2-1) pendant vertices with first vertex and a_2-2 pendant vertices with next $m_2 - 1$ vertices. Now add $a_i - 2$ pendant vertices to next m_i vertices, for each $i \leq (n-1)$. For last m_n vertices, add $m_n - 1$ pendant vertices to the last vertex and for remaining ones add $m_n - 2$ pendant vertices. In this tree, first m_2 vertices are having degree a_2, next m_3 vertices are having degree a_3 and so on. So $|V| = \sum_{i=2}^{n} m_i+(a_2-1)+(m_2-1)(a_2-2)+\sum_{i=3}^{n-1} m_i(a_i-2) + (a_n - 1) + (m_n - 1)(a_n - 2) = \sum_{i=2}^{n} m_i(a_i - 1) + 2$.