Deniable Fully Homomorphic Encryption from LWE

Shweta Agrawal, Shafi Goldwasser, Saleet Mossel

Crypto, 2021 (To appear)

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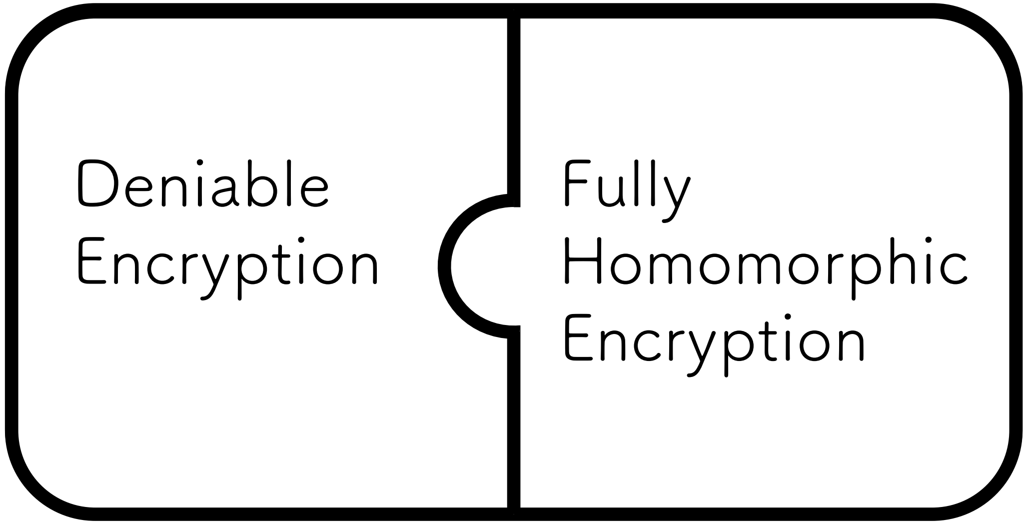
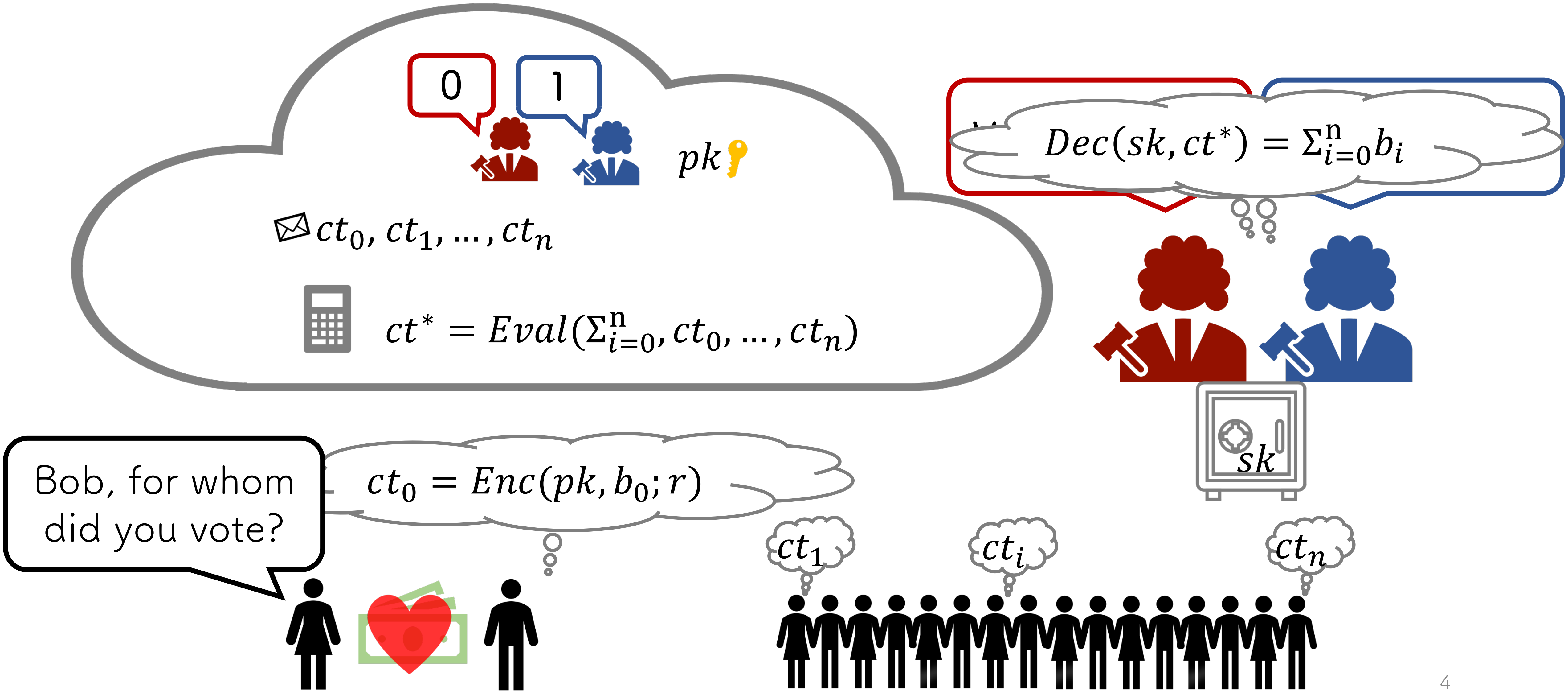
Most slides by Saleet Mossel



Deniable FHE

The notion of Deniable FHE

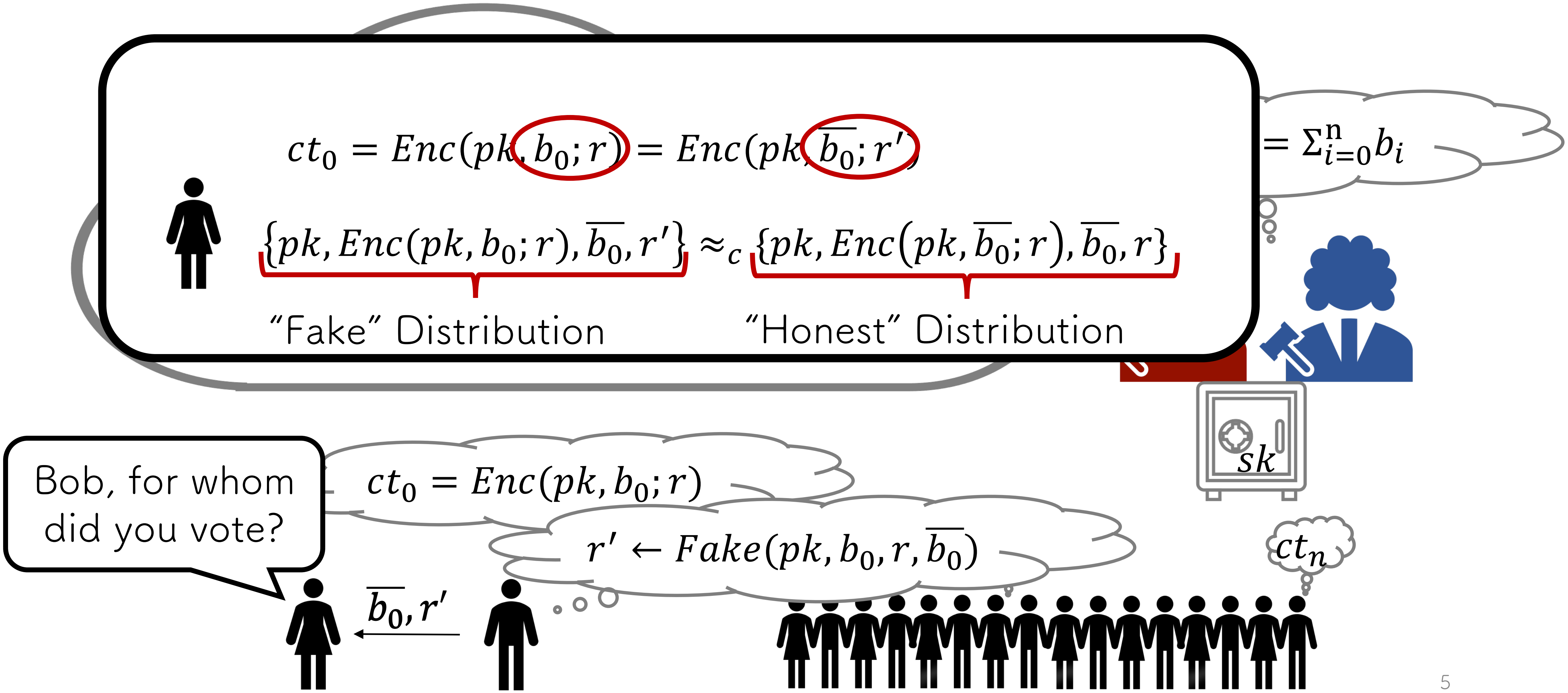
Deniable FHE



Deniable FHE

$$ct_0 = Enc(pk, b_0; r) = Enc(pk, \overline{b_0}; r')$$

$$\underbrace{\{pk, Enc(pk, b_0; r), \overline{b_0}, r'\}}_{\text{“Fake” Distribution}} \approx_c \underbrace{\{pk, Enc\left(pk, \overline{b_0}; r\right), \overline{b_0}, r\}}_{\text{“Honest” Distribution}}$$



Elections require Deniability & FHE

- Benefit of Deniable Encryption in Elections:
  - Honest Participation

- Benefit of Fully Homomorphic Encryption in Elections:
  - Homomorphically compute the voting result

Any data-driven algorithm

Store Encrypted Data

Natural upgrade for all DE apps!

Deniable Encryption

- Introduced by **Canetti, Dwork, Naor and Ostrovsky 1997**
  - construction from trapdoor permutations, unique SVP
  - size of ct is the inverse of the detection probability

- Weak Deniable Encryption
  - can also lie about the encryption algorithm (Enc, Denc)
  - construction with compact ct and negligible deniability

- Lower bound (Efficiency vs. Deniability)
  - It **seems inherent** that the length of ct grows with the inverse of the detection probability in "separable" constructions.

- A significant step forward [SW14]
  - construction from iO and OWF
  - compact ct and negligible deniability

What does this mean given recent iO results?

Deniable Encryption

CDNO

- Based on TDP
- CT size inverse of detection prob

SW

- Based on iO
- CT size indpt of detection prob

1997

2014

In full model, nothing else known!

Our Results

- Notion of Deniable FHE (full and weak)
- Constructions based on Learning With Errors
- Compact ct : size does not depend on detection probability!
  - Our construction is separable (so not inherent)
  - Total encryption time grows with the inverse of the detection probability!
- Support large message space
  - All prior work encode large messages bit by bit
- Offline-Online Encryption
  - Online time independent of the detection probability

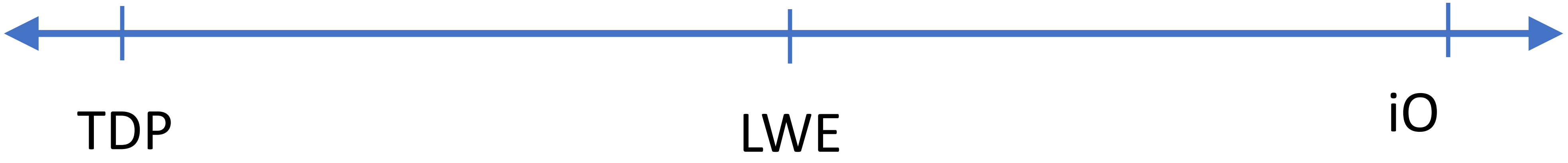
Our Results: Deniable Encryption

CDNO, 1997

CT size inverse of detection prob

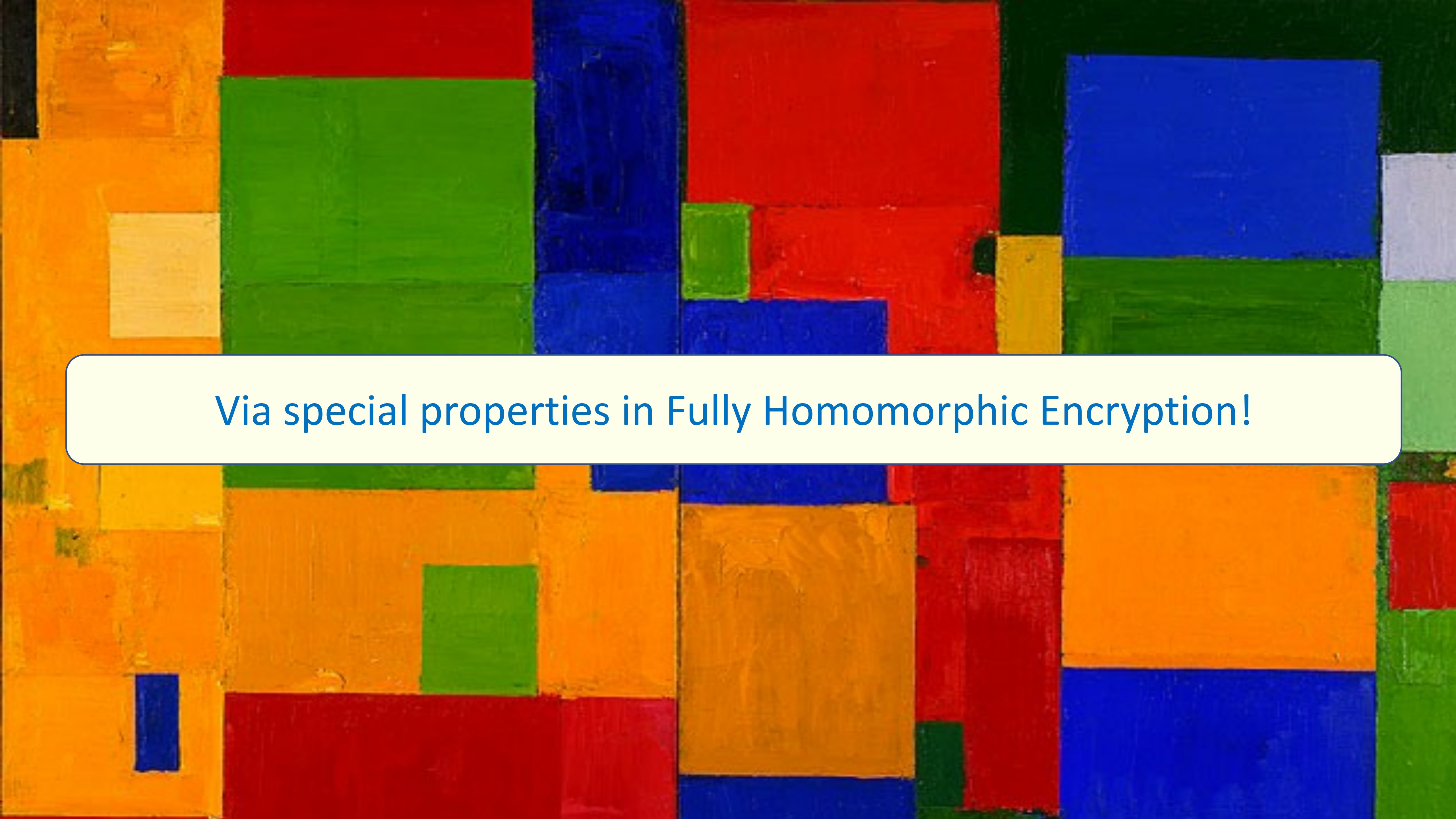
SW, 2014

CT size indpt of detection prob



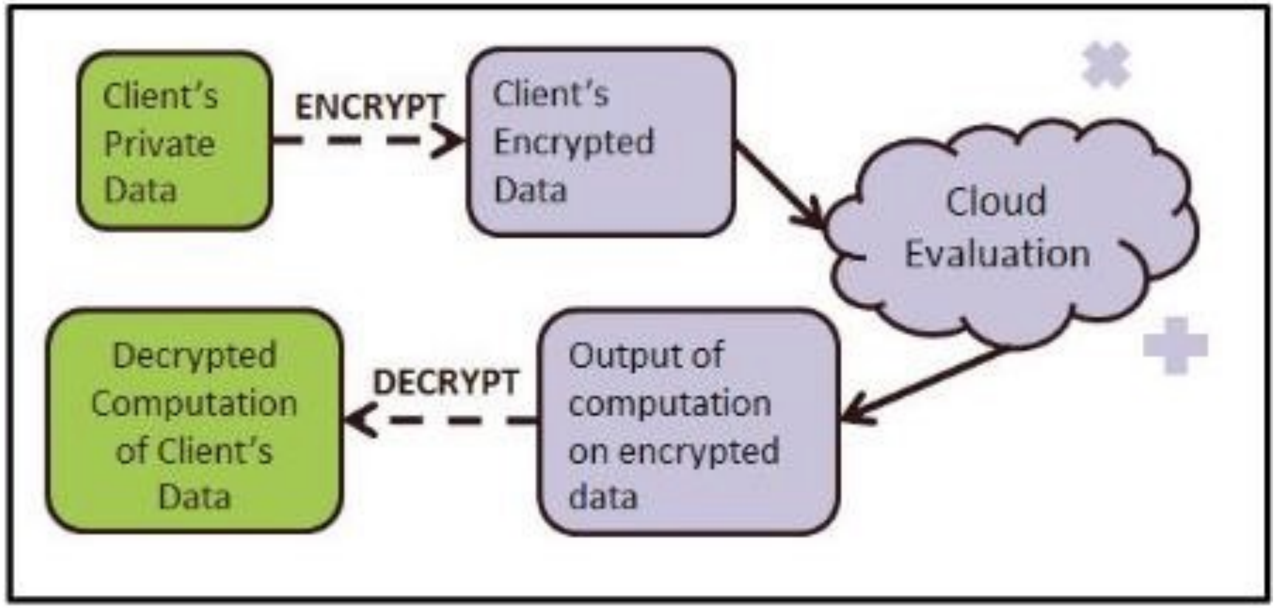
This Work

- CT size independent of detection prob
- (Offline) encryption time inverse of detection prob


Via special properties in Fully Homomorphic Encryption!

Fully Homomorphic Encryption

Can be built using LWE (BV11, BGV12, GSW13…)



Expressive Functionality: Supports arbitrary circuits

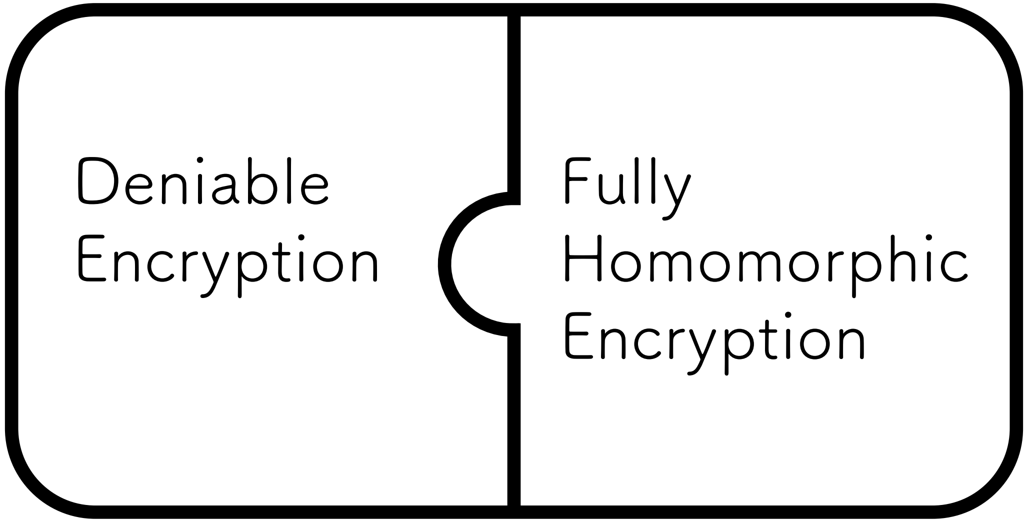
Compact ciphertext, independent of circuit size

Encryption and function evaluation commute!
Enc(f(x)) =* f(Enc(x))

* : roughly

Adding Deniability to the Mix

- A Deniable FHE scheme $(Gen, Enc, Eval, Dec, Fake)$
  - $(Gen, Enc, Eval, Dec)$ is an FHE scheme
  - $(Gen, Enc, Dec, Fake)$ is a Deniable Encryption scheme



Deniable FHE

A Deniable FHE scheme $(Gen, Enc, Eval, Dec, Fake)$ syntax

- $Gen \rightarrow (pk, sk)$
- $Enc(pk, m; r) = ct$
- $Dec(sk, ct) = b$
- $Eval(pk, f, ct_1, \dots, ct_k) = ct^*$
- $Fake(pk, b, r, \bar{b}) \rightarrow r'$

Deniable FHE

A Deniable FHE scheme $(Gen, Enc, Eval, Dec, Fake)$

1. Correctness
2. CPA-Security
3. Deniability
4. Compactness

Correctness versus Deniability

Correctness:

For every $\boldsymbol{f}$ and $\boldsymbol{m_1, \dots, m_k}$:

$$\mathbf{Pr}\left[\boldsymbol{Dec}\big(\boldsymbol{sk}, \boldsymbol{Eval}(\boldsymbol{pk}, \boldsymbol{f}, \boldsymbol{ct_1}, \dots, \boldsymbol{ct_k})\big) = \boldsymbol{f}(\boldsymbol{m_1}, \dots, \boldsymbol{m_k})\right] = \mathbf{1} - \boldsymbol{negl}$$

where $\boldsymbol{ct_i} \leftarrow \boldsymbol{Enc}(\boldsymbol{pk}, \boldsymbol{m_i})$ and $(\boldsymbol{pk}, \boldsymbol{sk}) \leftarrow \boldsymbol{Gen}$

Cannot simultaneously satisfy <u>perfect</u> correctness and <u>deniability</u>

$\delta(\lambda)$- Deniability

We consider (inverse) polynomial deniability

For every bit $\boldsymbol{b}$, and PPT adversary $\boldsymbol{A}$

detection probability

$$\left|\mathbf{Pr}\big[\boldsymbol{A}\underbrace{(\boldsymbol{pk},\boldsymbol{Enc}(\boldsymbol{pk},\boldsymbol{b};\boldsymbol{r}),\boldsymbol{b},}_{\text{“Honest” Distribution}}\boldsymbol{r})\big]-\mathbf{Pr}\big[\boldsymbol{A}\big(\underbrace{\boldsymbol{pk},\boldsymbol{Enc}(\boldsymbol{pk},\overline{\boldsymbol{b}};\boldsymbol{r}),\boldsymbol{b},\boldsymbol{r}'}_{\text{“Fake” Distribution}}\big)\big]\right|\leq\boldsymbol{\delta}(\boldsymbol{\lambda})$$

where $(\boldsymbol{pk},\boldsymbol{sk})\leftarrow\boldsymbol{Gen}$, $\boldsymbol{r}\leftarrow\{\mathbf{0},\mathbf{1}\}^{\ell'}$, and $\boldsymbol{r}'\leftarrow\boldsymbol{Fake}(\boldsymbol{pk},\overline{\boldsymbol{b}},\boldsymbol{r},\boldsymbol{b})$

Evaluation & Deniability Compactness

a) For every $\boldsymbol{f}$ and $\boldsymbol{m_1, \ldots, m_k}$:

$$|\boldsymbol{Eval}(\boldsymbol{pk}, \boldsymbol{f}, \boldsymbol{ct_1}, \ldots, \boldsymbol{ct_k})| \leq \boldsymbol{poly}$$

where $\boldsymbol{ct_i} \leftarrow \boldsymbol{Enc}(\boldsymbol{pk}, \boldsymbol{m_i})$ and $(\boldsymbol{pk}, \boldsymbol{sk}) \leftarrow \boldsymbol{Gen}$

Independent of k and the complexity of f

b) For every $\boldsymbol{m}$:

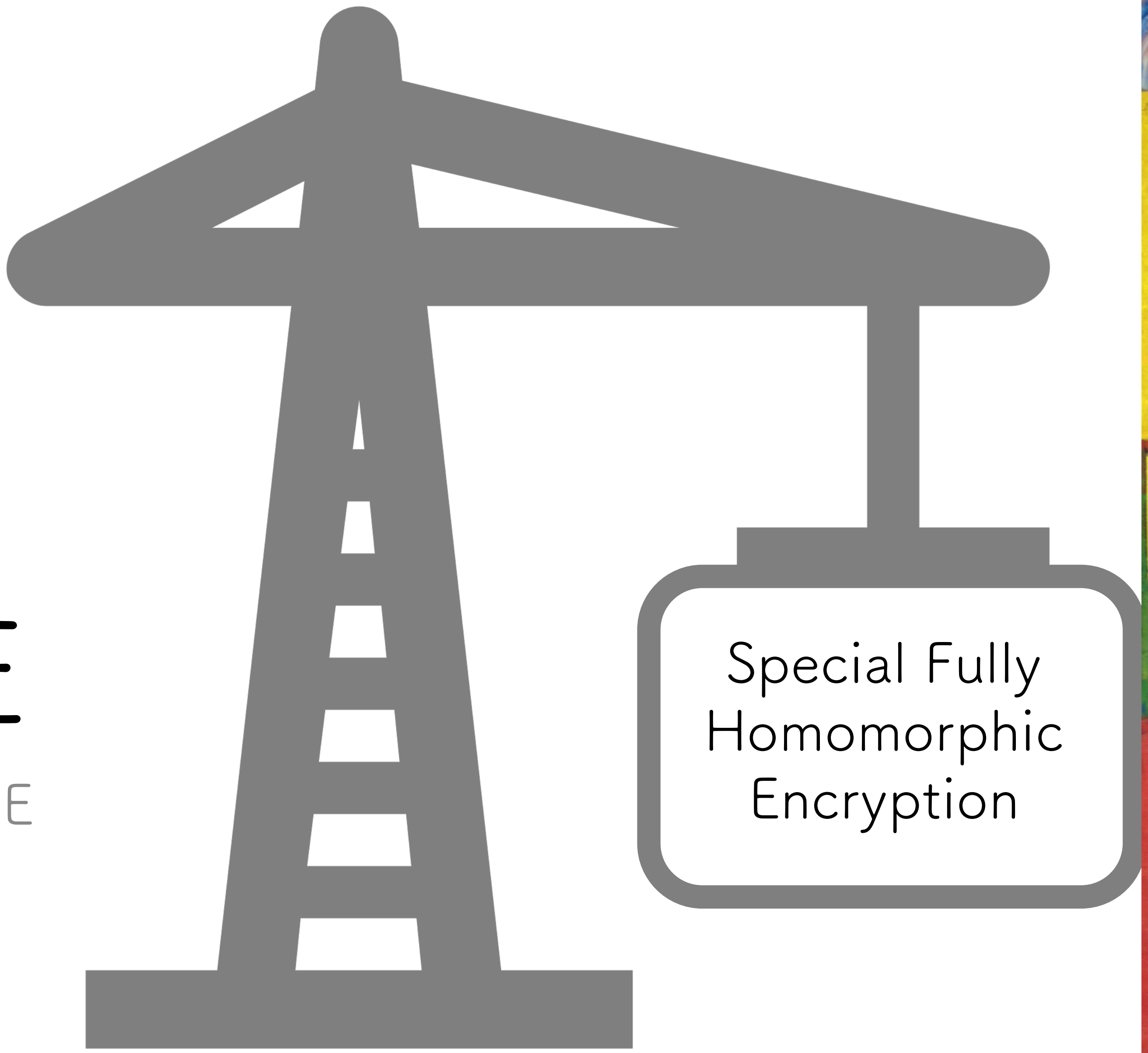
$$|\boldsymbol{Enc}(\boldsymbol{pk}, \boldsymbol{m})| \leq \boldsymbol{poly}$$

where $(\boldsymbol{pk}, \boldsymbol{sk}) \leftarrow \boldsymbol{Gen}$, regardless of the encryption running time

Independent of the detection probability

Deniable FHE

Our Construction of Deniable FHE



FHE from LWE: A Very Brief Recap

- All* known FHE schemes add noise in CT for security.
- Homomorphic evaluation of CTs ($\text{eval}(f, ct_1 \cdots ct_n)$) cause noise to grow
- Kills correctness after noise grows too much
- Limits number of homomorphic operations

How to keep going: Gentry's bootstrapping [Gen09]!

The Magic of Bootstrapping

- Assume that an FHE is powerful enough to support evaluation of its own decryption circuit Dec.

- By correctness of decryption, Dec(ct_x, sk) = x

$$\mathrm{Dec}\left(\boxed{x}, sk\right) = x$$

- Define circuit $\mathrm{Dec}_{ct}(sk) = \mathrm{Dec}(sk, ct)$
- By correctness of homomorphic evaluation, Eval(F, ct_x) = ct(F(x))

$$\mathrm{Eval}\left(\mathrm{Dec}_{ct}, \boxed{sk}\right) = \boxed{\mathrm{Dec}_{ct}(sk)} = \boxed{x}$$

The Magic of Bootstrapping

- Originally introduced to reduce noise in evaluated ciphertext

- Homomorphic evaluation of decryption
  - removes large old noise
  - adds small new noise (size small since decryption shallow)

This work: Oblivious Sampling of FHE ciphertexts!

The Magic of Bootstrapping

- Assume that decryption <u>always</u> outputs 0 or 1
  - even if input ct is not well formed

- Then, bootstrapping <u>always</u> outputs proper encryption of 0 or 1!

$$\mathrm{Eval}\left(\mathrm{Dec}_{ct}, \boxed{sk}\right) = \boxed{\mathrm{Dec}_{ct}(sk)} = \boxed{x}$$

Even if input “ct” is a random element in ciphertext space!

The Magic of Bootstrapping

- Assume that decryption <u>outputs 0 w.o.p for random input</u>

- Then, bootstrapping outputs <u>encryption of 0 w.o.p for random input</u>

$$\text{Eval}\left(\text{Dec}_{\text{rand}}, \boxed{\text{sk}}\right) = \boxed{\text{Dec}_{\text{rand}}(\text{sk})} = \boxed{0}$$

Given enc(sk), run dec homomorphically on random to generate encryption of 0 w.o.p!

But, wait a minute…

- Given encryption of 1, decryption outputs 1 w.o.p
- Encryption of 1 is indistinguishable from random!

$$\mathrm{Eval}\left(\mathrm{Dec}_{ct1},\ \boxed{sk}\right) = \boxed{\mathrm{Dec}_{ct1}(sk)} = \boxed{1}$$

- Can pretend as if ct1 = enc(1) is a random string

Pretend bootstrapping outputs enc(0) but actually enc(1)!

Can provide randomness R so it looks like Bootstrap(R) = enc(0) but actually enc(1)

OK… but why is this useful?

Leveraging our trick (binary msg space)

- Let $B(x) = Eval(pk, Dec_x, ct_{sk})$ the bootstrapping procedure
  - recall $Dec_x(sk) = Dec(sk, x)$

- Denote homomorphic addition (mod 2) as

$$Eval(pk, +, ct_a, ct_b) = ct_a \oplus ct_b$$

$B(R_i)$
$= Enc(x_i)$

$$B(R_1) \oplus \cdots \oplus B(R_n) = \text{Enc}\left(\text{Parity}\left(x_1, \ldots, x_n\right)\right)$$

Construction

Gen:

1. $(pk, sk) \leftarrow Gen$
2. $ct_{sk} \leftarrow Enc(pk, sk)$
3. Output $pk = (pk, ct_{sk}), sk = sk$

Construction

$rand = (x_1, \dots, x_n, \{R_i\}_{x_i=0}, \{r_i\}_{x_i=1})$

$Enc(pk, b)$:

1. Sample $x_1, \dots, x_n \leftarrow \{0,1\}$ s.t. $\sum_i x_i = b\ (mod\ 2)$
2. For $x_i = 0$, sample $R_i \leftarrow \mathcal{R}^\ell$
3. For $x_i = 1$, sample $r_i \leftarrow \{0,1\}^{\ell'}$ and set $R_i = Enc(pk, 1; r_i)$
4. Compute $ct = B(R_1) \oplus \cdots \oplus B(R_n)$
5. Output ct

$B(\mathcal{R}^\ell)$ is a valid encryption of 0 w.h.p

Construction

$rand = (x_1, \dots, x_n, \{R_i\}_{x_i=0}, \{r_i\}_{x_i=1})$

$Fake(pk, b, rand, \bar{b})$:

1. If $b = \bar{b}$, output $rand$
2. Sample $k \leftarrow [n]$ s.t. $x_k = 1$
3. Set $x'_k = 0$ and $R'_k = Enc(pk, 1; r_k)$
4. For $i \neq k$, set $R'_i = R_i$ and $r'_i = r_i$
5. Output $rand' = (x'_1, \dots, x'_n, \{R'_i\}_{x'_i=0}, \{r'_i\}_{x'_i=1})$

Pseudorandom Ciphertext

By pretending one ciphertext enc(1) is random, parity flipped!

Statistical distance from honest dist is 1/poly(n)

Construction

$Eval(pk, f, ct_1, \dots, ct_k)$:

1. Interpret ct_i as special FHE ciphertext ct_i
2. Output $Eval(pk, f, ct_1, \dots, ct_k)$

$Dec(dsk, ct)$:

1. Interpret ct as special FHE ciphertext ct
2. Output $Dec(sk, ct)$

As before!

Deniable FHE

Proof of Correctness, CPA-Security, Compactness, Deniability

Proof: Correctness

- The output is a ciphertext of the Special FHE.
- If with high probability $B(\mathcal{R}^\ell)$ is a valid encryption of 0, then with high probability $Enc(pk, b)$ is a valid encryption of b.

$Enc(pk, b)$:

1. Sample $x_1, \dots, x_n \leftarrow \{0,1\}$ s.t. $\sum_i x_i = b \ (mod\ 2)$
2. For $x_i = 0$, sample $R_i \leftarrow \mathcal{R}^\ell$
3. For $x_i = 1$, sample $r_i \leftarrow \{0,1\}^{\ell'}$ and set $R_i = Enc(pk, 1; r_i)$
4. Compute $ct = B(R_1) \oplus \cdots \oplus B(R_n)$
5. Output ct

1. Correctness

   For every $\boldsymbol{f}$ and $\boldsymbol{m_1, \dots, m_k}$:

   $$\mathbf{Pr}[\boldsymbol{Dec(sk, Eval(pk, f, ct_1, \dots, ct_k)) = f(m_1, \dots, m_k)}] = \mathbf{1} - \boldsymbol{negl}$$

   where $\boldsymbol{ct_i} \leftarrow \boldsymbol{Enc(pk, m_i)}$ and $\boldsymbol{(pk, sk)} \leftarrow \boldsymbol{Gen}$

Proof: CPA-Security

- The output is a ciphertext of the Special FHE.
- The public key is (pk, ct_{sk})
- If the special FHE is circular secure, then the scheme is secure.

2. CPA-Security

$$\{\boldsymbol{pk}, \boldsymbol{Enc}(\boldsymbol{pk}, \mathbf{0})\} \approx_c \{\boldsymbol{pk}, \boldsymbol{Enc}(\boldsymbol{pk}, \mathbf{1})\}$$

where $(\boldsymbol{pk}, \boldsymbol{sk}) \leftarrow \boldsymbol{Gen}$

Gen:

1. $(pk, sk) \leftarrow Gen$
2. $ct_{sk} \leftarrow Enc(pk, sk)$
3. Output $pk = (pk, ct_{sk}), sk = sk$

Proof: $\delta(\lambda)$-Deniability

- First, prove that $Enc(pk, \bar{b}; r) = Enc(pk, b, r')$.
- We can remove the ciphertext from A's input.
  - It is a function of A's input.
- Last, prove the distance is $\delta(\lambda)$

3. $\delta(\lambda)$-Deniability

For every bit $\boldsymbol{b}$, and PPT adversary $\boldsymbol{A}$

$$|\mathbf{Pr}[\boldsymbol{A}(\boldsymbol{pk}, \boldsymbol{b}, \boldsymbol{r})] - \mathbf{Pr}[\boldsymbol{A}(\boldsymbol{pk}, \boldsymbol{b}, \boldsymbol{r}')]| \leq \boldsymbol{\delta}(\boldsymbol{\lambda})$$

where $(\boldsymbol{pk}, \boldsymbol{sk}) \leftarrow \boldsymbol{Gen}$, $\boldsymbol{r} \leftarrow \{\mathbf{0}, \mathbf{1}\}^{\ell'}$, and $\boldsymbol{r}' \leftarrow \boldsymbol{Fake}(\boldsymbol{pk}, \bar{\boldsymbol{b}}, \boldsymbol{r}, \boldsymbol{b})$

Proof: $\delta(\lambda)$-Deniability

- Prove that $Enc(pk, \bar{b}; r) = Enc(pk, b, r')$
  - uniform r and $r' \leftarrow Fake(pk, \bar{b},$
- Real: $r = x_1, \dots, x_n, \{R_i\}_{x_i=0}, \{r_i\}$
  - r is uniform conditioned on $\sum x_i$
- Fake: $r' = x_1', \dots, x_n', \{R_i'\}_{x_i'=0}, \{r_i'\}_{x_i'=1}$
  - r' is equal to r except:
    - $x_k' = \overline{x_k} = 0$ and $R_k' = Enc(pk, 1; r_k)$

$Enc(pk, b)$:
1. Sample $x_1, \dots, x_n \leftarrow \{0,1\}$ s.t. $\sum_i x_i = b \ (mod\ 2)$
2. For $x_i = 0$, sample $R_i \leftarrow \mathcal{R}^{\ell}$
3. For $x_i = 1$, sample $r_i \leftarrow \{0,1\}^{\ell'}$ and set $R_i = Enc(pk, 1; r_i)$
4. Compute $ct = B(R_1) \oplus \cdots \oplus B(R_n)$
5. Output ct

$Fake(pk, b, rand, \bar{b})$:
1. If $b = \bar{b}$, output $rand$
2. Sample $k \leftarrow [n]$ s.t. $x_k = 1$
3. Set $x_k' = 0$ and $R_k' = Enc(pk, 1; r_k)$
4. For $i \neq k$, set $R_i' = R_i$ and $r_i' = r_i$
5. Output $rand' = (x_1', \dots, x_n', \{R_i'\}_{x_i'=0}, \{r_i'\}_{x_i'=1})$

$\sum x_i' = b \ (mod\ 2)$

Output is identical

Proof: $\delta(\lambda)$-Deniability

- Last, prove the distance is $\delta(\lambda)$

- If special FHE has **pseudorandom ciphertext**, then the following are computational indistinguishable

  - Fake $r' = x'_1, \dots, x'_n, \{R'_i\}_{x'_i=0}, \{r'_i\}_{x'_i=1}$ s.t.
    - $R'_k = Enc(pk, 1; r_k)$ and $r_k \leftarrow \{0,1\}^{\ell'}$

  - Mid $r' = x'_1, \dots, x'_n, \{R'_i\}_{x'_i=0}, \{r'_i\}_{x'_i=1}$ s.t.
    - $R'_k \leftarrow \mathcal{R}^{\ell}$

Proof: $\delta(\lambda)$-Deniability

- Last, prove the distance is $\delta(\lambda)$

set $n = \frac{1}{\delta(\lambda)^2}$

- The Statistical Distance of the following two distributions is $\frac{1}{\sqrt{n}}$
  - Mid $r' = x_1', \dots, x_n', \{R_i'\}_{x_i'=0}, \{r_i'\}_{x_i'=1}$ s.t.
    - Sample $x_1, \dots, x_n \leftarrow \{0,1\}$ such that $\sum x_i = \bar{b} \ (mod\ 2)$
    - Sample $k \leftarrow [n]$ such that $x_i = 1$
    - Set $x_k' = 0$ and for $i \neq \mathrm{k}$ set $x_i' = x_i$

$\mathrm{E}[\sum x_i] > \mathrm{E}[\sum x_i']$

  - Real $r = x_1, \dots, x_n, \{R_i\}_{x_i=0}, \{r_i\}_{x_i=1}$ s.t.
    - Sample $x_1, \dots, x_n \leftarrow \{0,1\}$ such that $\sum x_i = b \ (mod\ 2)$

Proof: Compactness

- The output is a ciphertext of the Special FHE.

4. Compactness

   a) For every $\boldsymbol{f}$ and $\boldsymbol{m_1, \dots, m_k}$:

   $$|\boldsymbol{Eval(pk, f, ct_1, \dots, ct_k)}| \le \boldsymbol{poly}$$

   Independent of k and the complexity of f

   where $\boldsymbol{ct_i} \leftarrow \boldsymbol{Enc(pk, m_i)}$ and $\boldsymbol{(pk, sk)} \leftarrow \boldsymbol{Gen}$

   b) For every $\boldsymbol{m}$:

   $$|\boldsymbol{Enc(pk, m)}| \le \boldsymbol{poly}$$

   Independent of the faking probability

   where $\boldsymbol{(pk, sk)} \leftarrow \boldsymbol{Gen}$, regardless of the encryption running time

Deniability Compactness from Evaluation Compactness!

Special FHE

Definition and Instantiation

Special FHE

Circular security

Pseudo-random CT

Det. eval and dec

$B(\mathcal{R}^{\ell})$ is a valid encryption of 0 w.h.p

Can be removed

Usually OK

Can be weakened whp to wnnp

Weaker Special FHE

1. Pseudorandom Ciphertext
2. Deterministic evaluation and decry
3. Decryption always outputs a valid mess and

Almost always the case

$$\mathbf{Pr}[\boldsymbol{Dec}(\boldsymbol{sk},\boldsymbol{R}) = \mathbf{0}] = \mathbf{1}/\boldsymbol{poly}$$

where $\boldsymbol{R} \leftarrow \mathcal{R}^{\ell}$ and $(\boldsymbol{pk},\boldsymbol{sk}) \leftarrow \boldsymbol{Gen}$

[BGV14] FHE satisfies all properties!

Instantiation of Special FHE

- In [BGV14] given the sk one can check if ct is well-formed
- We modify the decryption algorithm of [BGV14]:

If well-formed:
then, output $Dec(sk, ct)$,
else output 0

$$Dec(sk, ct) = \left[[\langle sk, ct \rangle]_q \right]_2$$

Ciphertexts:

$$[\langle sk, ct \rangle]_q = b + 2e$$

where $|e| < B$

Random elements:

$$[\langle sk, R \rangle]_q = b + 2e$$

where $\Pr[|e| < B] = \frac{B}{q}$

Set q to be super polynomial, then $\frac{B}{q}$ is negligible

Online-Offline Encryption

Bulk of the computation is independent of the message, and may be performed in an offline pre-processing phase.

$Enc(dpk, b)$:

1. Select $x_1, \ldots, x_n \leftarrow \{0,1\}$ s.t. $\sum_i x_i = b \ (mod\ 2)$
2. For $x_i = 0$, select $R_i \leftarrow \mathcal{R}^\ell$
3. For $x_i = 1$, select $r_i \leftarrow \{0,1\}^{\ell'}$ and set $R_i = Enc(pk, 1; r_i)$
4. Output $dct = B(R_1) \oplus \cdots \oplus B(R_n)$

n-1 computations of $B(R_i)$ can be done offline:
choose R_n depending on b and compute $B(R_n)$ online

Main Takeaway:
Evaluation compactness in FHE implies deniability compactness in DE!

Going Forward

- Compact CT ➔ compact encryption runtime?
  - Analogy to FE [LPST16,GKPVZ13]
- Technical barrier: unidirectional cheating
- Need: Invertible oblivious sampling with bias
  - SW construction may be viewed through this lens
- From LWE: can have oblivious sampling with bias (this work) or oblivious sampling with inversion but not both (so far).

Thank You

Images Credit: Hans Hoffman