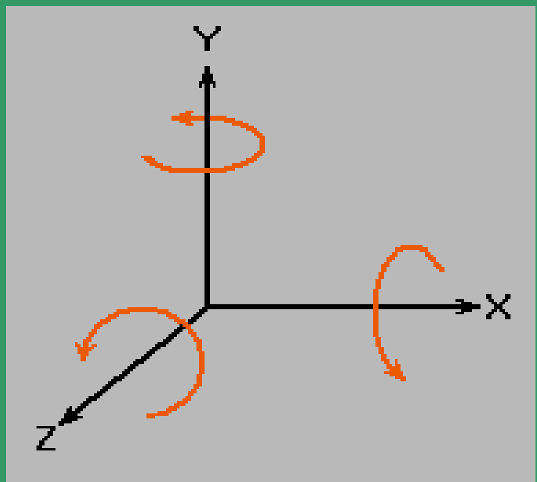
Three - Dimensional

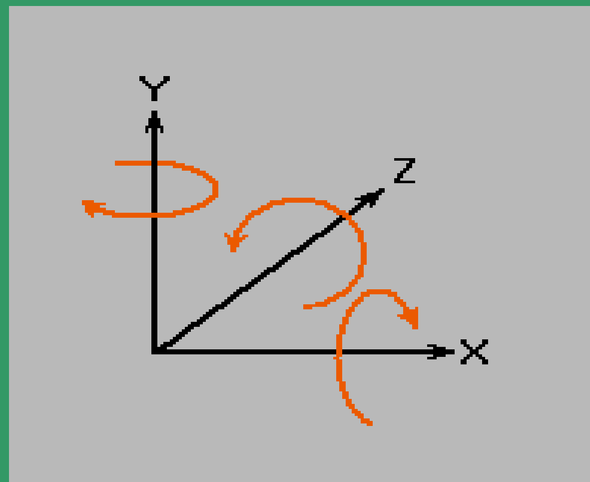
Graphics

Three-Dimensional Graphics

- **Use of a right-handed coordinate system (consistent with math)**
- **Left-handed suitable to screens.**
- **To transform from right to left, negate the z values.**



Right Handed Space



Left Handed Space

Homogeneous representation of a point in 3D space:

$$P = \left| \mathrm{x}\;\mathrm{y}\;\mathrm{z}\;\mathrm{w} \right|^{T}$$

$$(\mathrm{w} = 1, \text{for a 3D point})$$

Transformations will thus be represented by 4x4 matrices:

P' = A.P

Transformation Matrix in 3D:

$$A = \begin{bmatrix} a & b & c & p \\ d & e & f & q \\ g & i & j & r \\ l & m & n & s \end{bmatrix} = \begin{bmatrix} \mathrm{T} & \mathrm{K} \\ \Gamma & \Theta \end{bmatrix}$$

where,

$$\mathrm{T} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & i & j \end{bmatrix}$$

produces linear transformations: scaling, shearing, reflection and rotation.

$\mathrm{K} = [p\ q\ r]^T$, produces translation

$\Gamma = [l\ m\ n]^T$, yields perspective transformation

while, $\Theta = s$, is responsible for uniform scaling

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation

$$\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} 1 & 0 & Sh_x & 0 \\ 0 & 1 & Sh_y & 0 \\ 0 & Sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shear

Origin is unaffected by scale and shear

3D Reflection:

The following matrices:

$$T_{XY} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{YZ} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{ZX} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

produce reflection about:

XY plane **YZ plane** **ZX plane**

respectively.

Rotation Matrices along an axis:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

X-axis

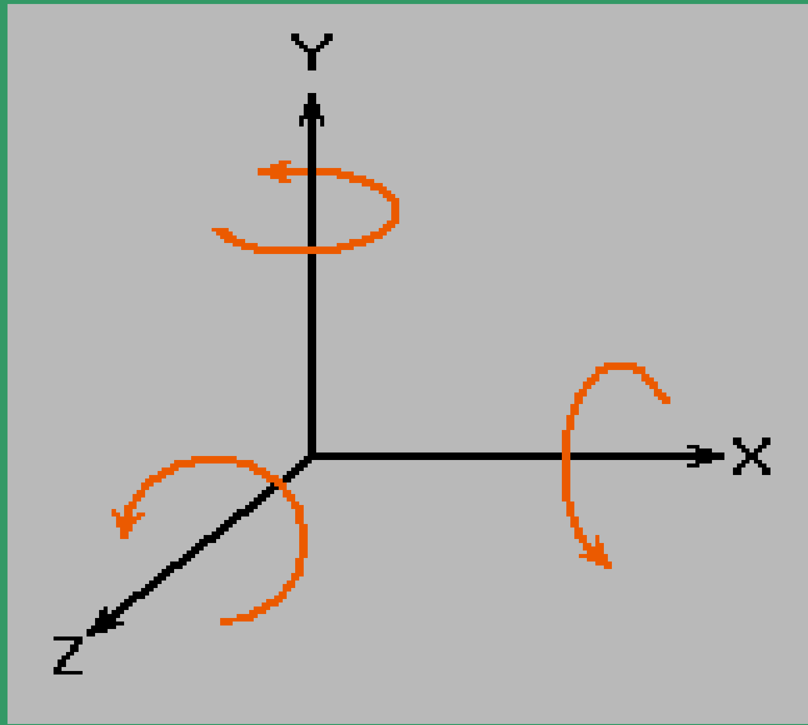
$$\begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

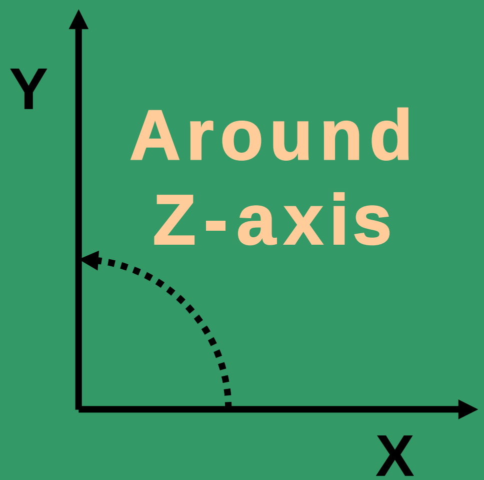
Y-axis

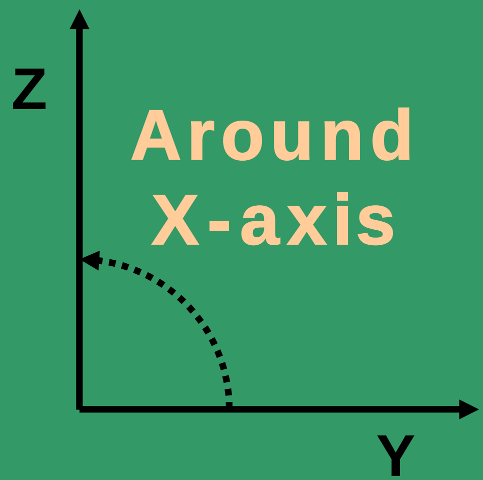
$$\begin{bmatrix} cos(\gamma) & -sin(\gamma) & 0 & 0 \\ sin(\gamma) & cos(\gamma) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

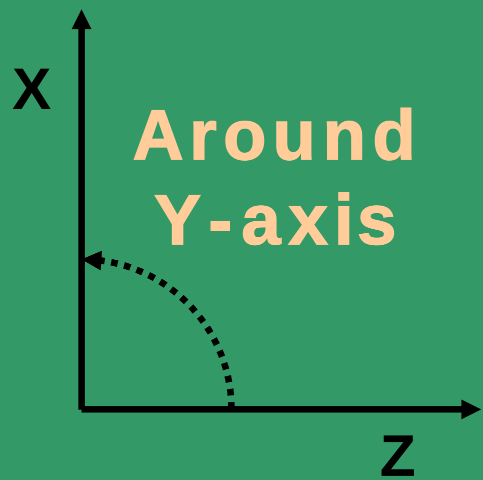
Z-axis

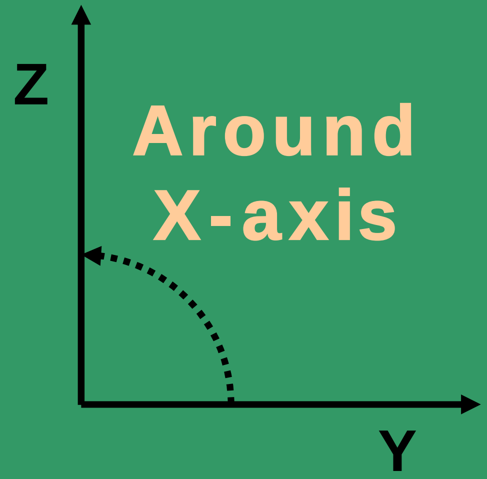
Why is the sign reversed in one case ?


Y
X
Z

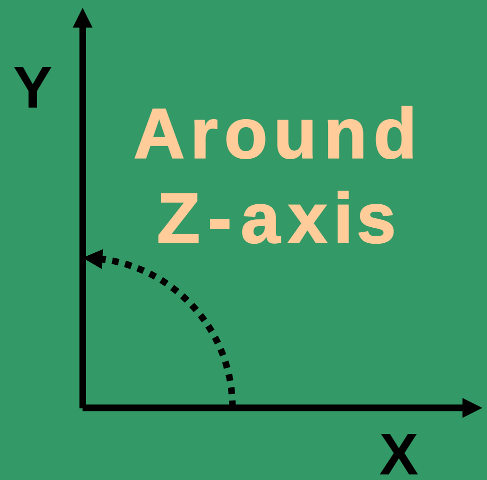

Y
Around
Z-axis
X


Z
Around
X-axis
Y

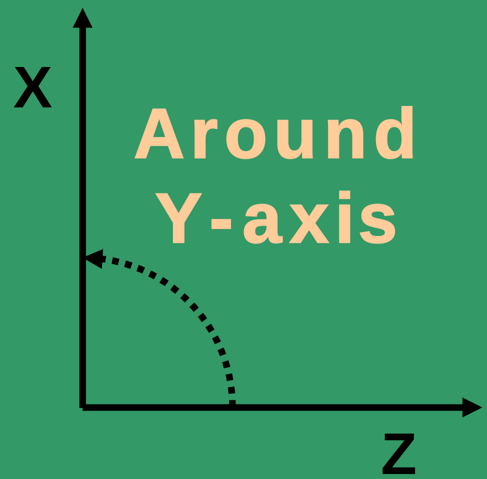

X
Around
Y-axis
Z



$$\begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \leftarrow X \\ \leftarrow Y \\ \leftarrow Z \\ \end{array} \end{array}$$



$$\begin{array}{cc} \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \leftarrow X \\ \leftarrow Y \\ \leftarrow Z \\ \end{array} \end{array}$$



$$\begin{array}{cc} \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \leftarrow X \\ \leftarrow Y \\ \leftarrow Z \\ \end{array} \end{array}$$

Rotation About an Arbitrary Axis in Space

Assume, we want to perform a rotation by θ degrees, about an axis in space passing through the point (x_0, y_0, z_0) with direction cosines (c_x, c_y, c_z).

1. First of all, translate by:
   $$|T| = -(x_0, y_0, z_0)^T$$
2. Next, we rotate the axis into one of the principle axes, let's pick, Z ($|R_x|$, $|R_y|$).
3. We rotate next by θ degrees in Z ($|R_z(\theta)|$).
4. Then we undo the rotations to align the axis.
5. We undo the translation: translate by $(-x_0, -y_0, -z_0)^T$

The tricky part of the algorithm is in step (2), as given before.

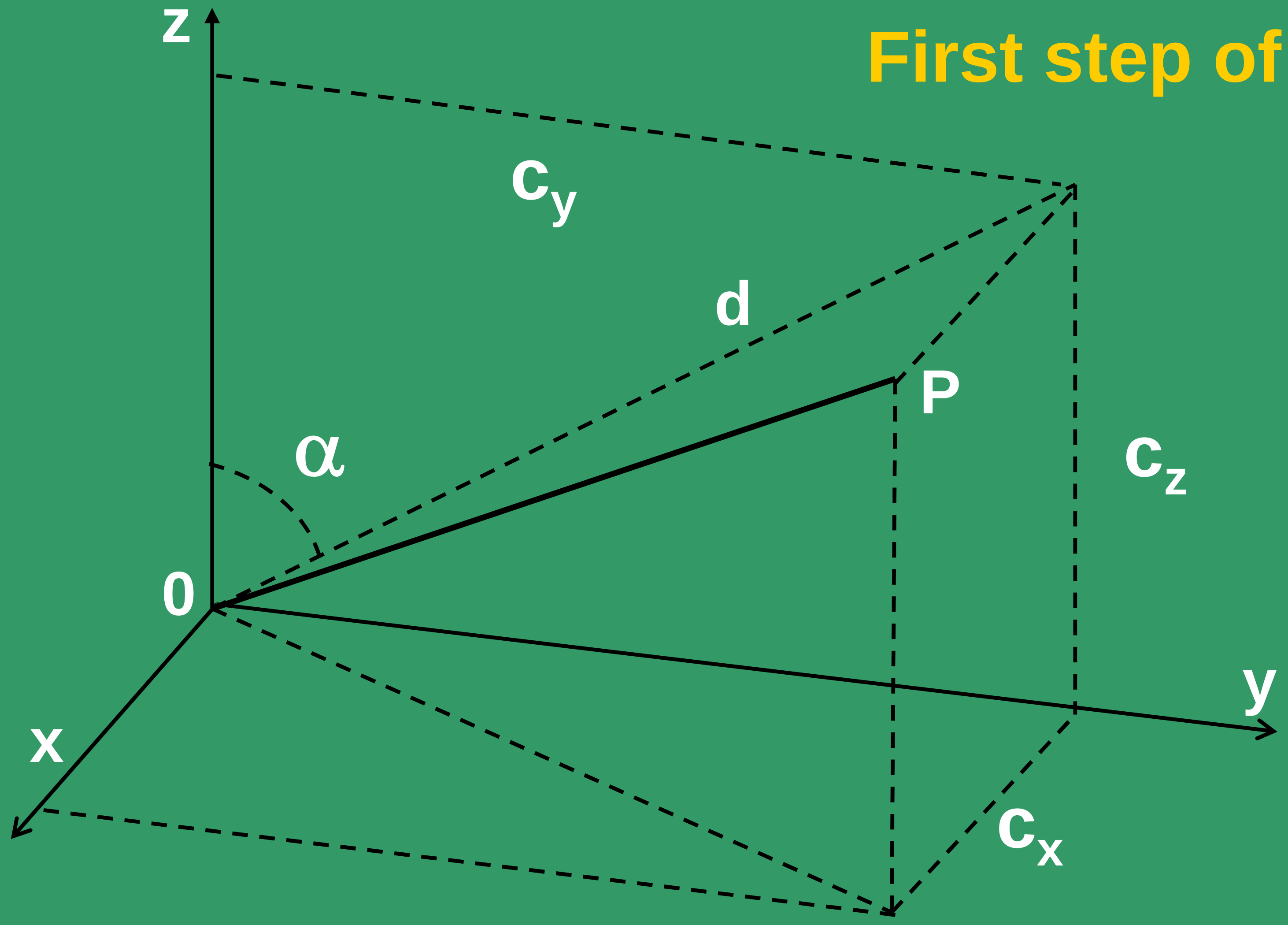
This is going to take 2 rotations:

i) About x-axis
(to place the axis in the xz plane)

and

ii) About y-axis
(to place the result coincident with the z-axis).

First step of Rotation:



Rotation about x by α:

How do we determine α?

Project the unit vector, along OP, into the yz plane.

The y and z components, c_y and c_z, are the direction cosines of the unit vector along the arbitrary axis.

It can be seen from the diagram, that :

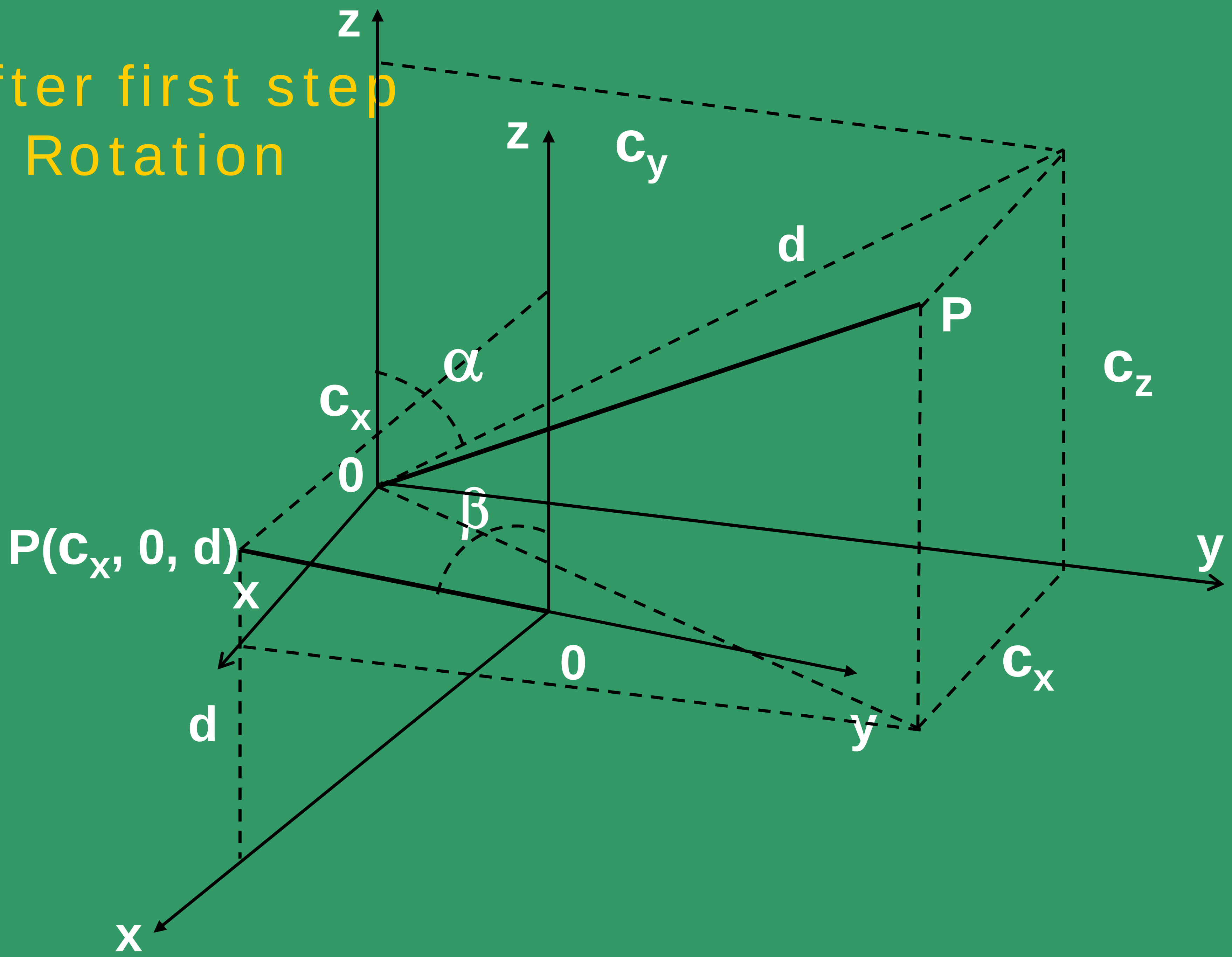
$$\mathrm{d} = \mathrm{sqrt}\left(C_y^2 + C_z^2\right)$$

$$\cos(\alpha) = {C_z}/{d}$$

$$\sin(\alpha) = {C_y}/{d}$$

$$\alpha = \sin^{-1}\left[\frac{c_y}{\sqrt{c_y^2 + c_z^2}}\right]$$

After first step of Rotation



Rotation by β about y:

How do we determine β?

Steps are similar to that done for α:

- Determine the angle β to rotate the result into the Z axis:
- The x component is c_x and the z component is d.

$\cos(\beta) = d = d / (\text{length of the unit vector})$

$\sin(\beta) = c_x = c_x / (\text{length of the unit vector})$.

Final Transformation for 3D rotation, about an arbitrary axis:

$$M = |T|\ |R_x|\ |R_y|\ |R_z|\ |R_y|^{-1}\ |R_x|^{-1}\ |T|^{-1}$$

Final Transformation matrix for 3D rotation, about an arbitrary axis:

$$M = |T|\ |R_x|\ |R_y|\ |R_z|\ |R_y|^{-1}\ |R_x|^{-1}\ |T|^{-1}$$

where:

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_z/d & -C_y/d & 0 \\ 0 & C_y/d & C_z/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$R_y = \begin{bmatrix} d & 0 & -C_x & 0 \\ 0 & 1 & 0 & 0 \\ C_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$M = |T|\ |R_x|\ |R_y|\ |R_z|\ |R_y|^{-1}\ |R_x|^{-1}\ |T|^{-1}$$

$$= [T\ R_x\ R_y]\ [R_z]\ [T\ R_x\ R_y]^{-1}$$

$$= C\ [R_z]\ C^{-1}$$

A special case of 3D rotation:

Rotation about an axis parallel to a coordinate axis (say, parallel to X-axis):

$$M_X = |T|\ \ |R_X|\ \ |T|^{-1}$$

Rotation About an Arbitrary Axis in Space

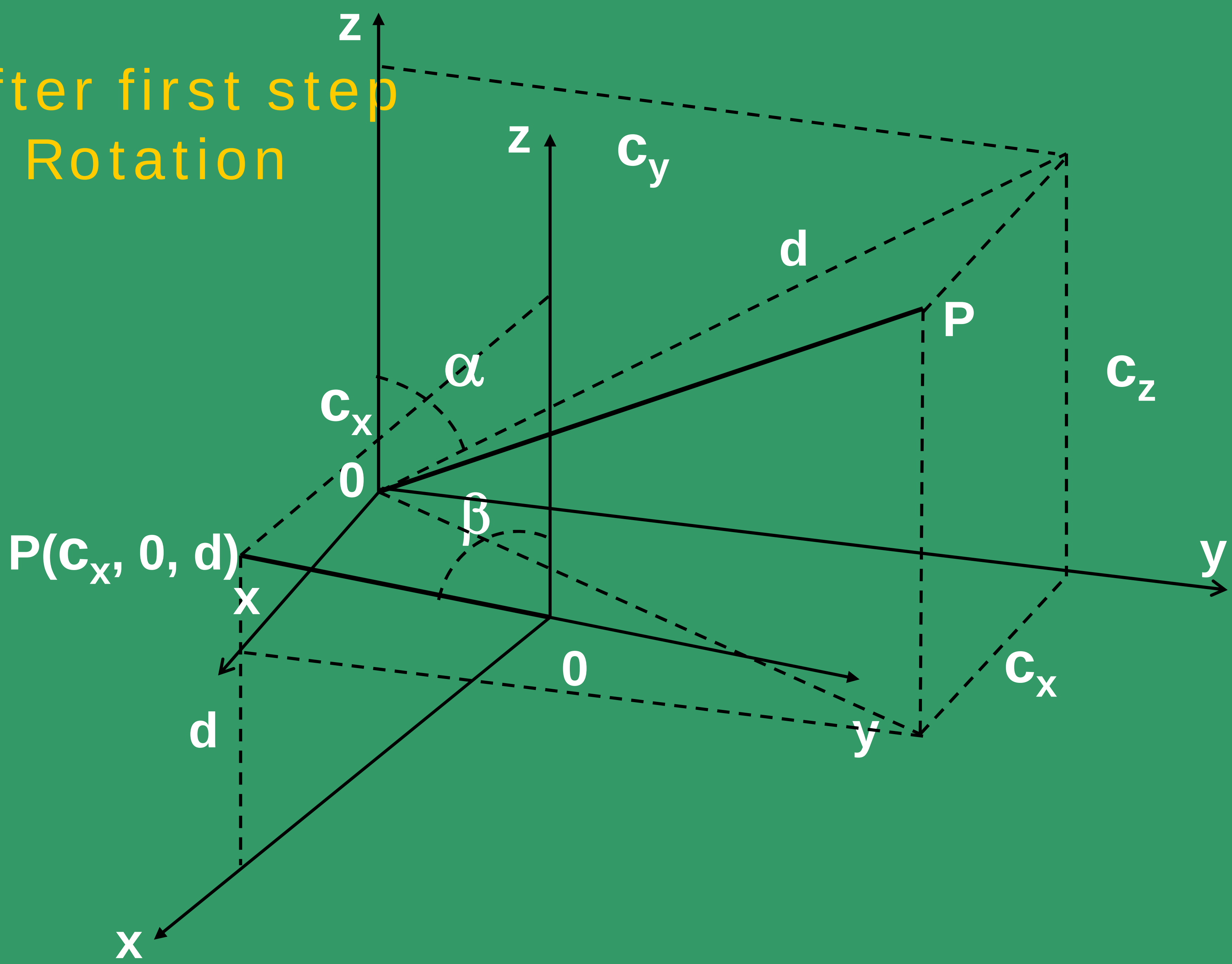
Assume, we want to perform a rotation by θ degrees, about an axis in space passing through the point (x_0, y_0, z_0) with direction cosines (c_x, c_y, c_z).

1. First of all, translate by:

   $$|T| = -(x_0, y_0, z_0)^T$$

2. Next, we rotate the axis into one of the principle axes, let's pick, Z ($|R_x|$, $|R_y|$).
3. We rotate next by θ degrees in Z ($|R_z(\theta)|$).
4. Then we undo the rotations to align the axis.
5. We undo the translation: translate by $(-x_0, -y_0, -z_0)^T$

After first step of Rotation



Final Transformation matrix for 3D rotation, about an arbitrary axis:

$$M = |T|\,|R_x|\,|R_y|\,|R_z|\,|R_y|^{-1}\,|R_x|^{-1}\,|T|^{-1}$$

where:

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_z/d & -C_y/d & 0 \\ 0 & C_y/d & C_z/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$R_y = \begin{bmatrix} d & 0 & -C_x & 0 \\ 0 & 1 & 0 & 0 \\ C_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

If you are given 2 points instead (on the axis of rotation), you can calculate the direction cosines of the axis as follows:

$$V = \left| \left(x_1 - x_0\right) \left(y_1 - y_0\right) \left(z_1 - z_0\right) \right|^T$$

$$c_x = \left(x_1 - x_0\right) / |V|$$

$$c_y = \left(y_1 - y_0\right) / |V|$$

$$c_z = \left(z_1 - z_0\right) / |V|,$$

where $|V|$ *is the lenght of the vector* V.

Reflection through an arbitrary plane

Method is similar to that of rotation about an arbitrary axis.

$$M = |T|\,|R_x|\,|R_y|\,|R_{fl}|\,|R_y|^{-1}\,|R_x|^{-1}\,|T|^{-1}$$

T does the job of translating the origin to the plane.

R_x and R_y will rotate the vector normal to the reflection plane (at the origin), until it is coincident with the +Z axis.

R_{fl} is the reflection matrix about X-Y plane or Z=0 plane.

Spaces

Object Space:

definition of objects. Also called Modeling space.

World Space:

where the scene and viewing specification is made

***Eyespace* (Normalized Viewing Space):**

where eye point (COP) is at the origin looking down the Z axis.

3D Image Space:

A 3D Projective space.

Dimensions: [-1:1] **in X & Y,** [0:1] **in Z.**

This is where image space hidden surface algorithms work.

Screen Space (2D):

Range of Coordinates -
[0 : width], [0 : height]

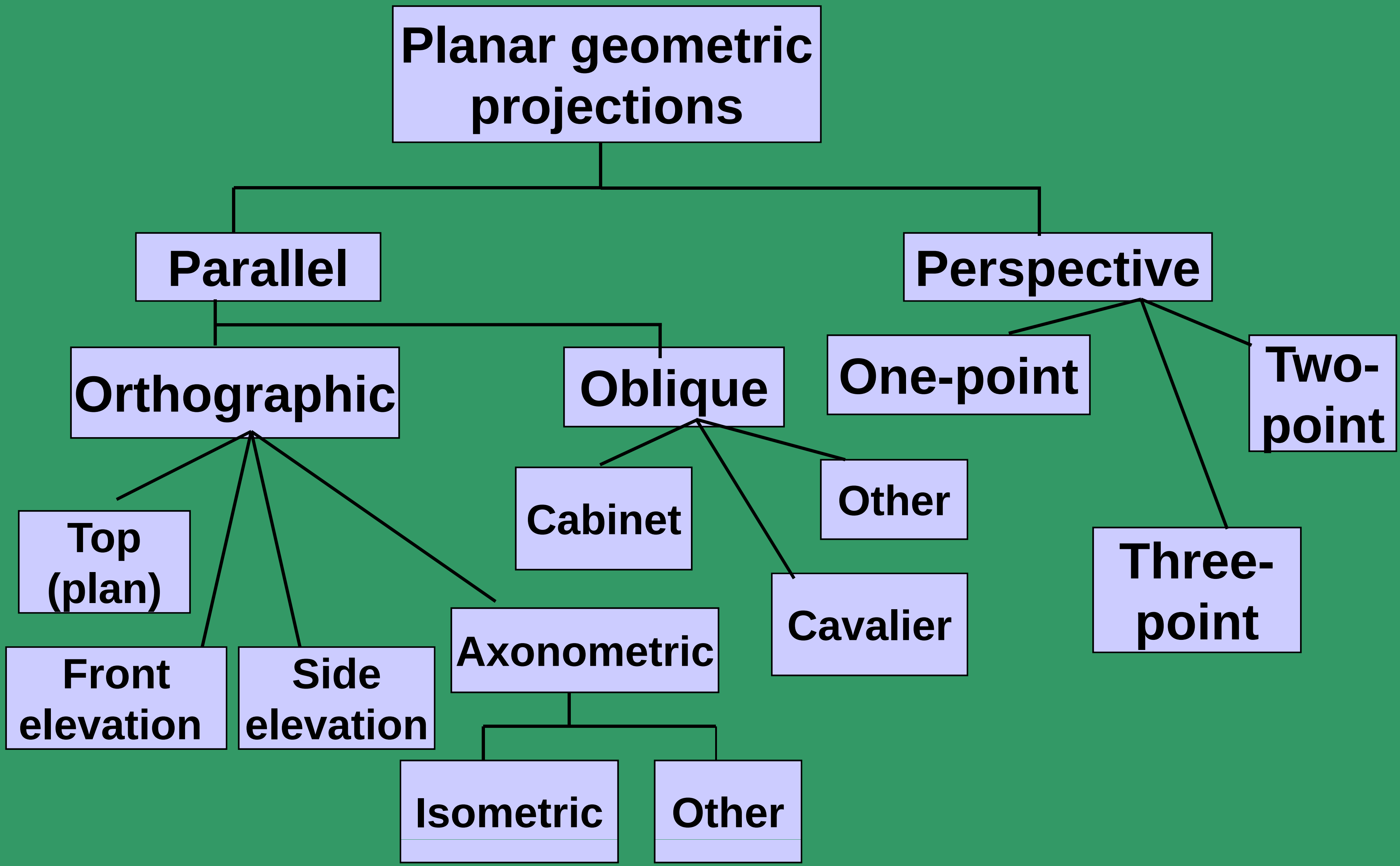
Projections

We will look at several planar geometric 3D to 2D projection:

- Parallel Projections
  - Orthographic
  - Oblique
- Perspective

Projection of a 3D object is defined by straight projection rays (projectors) emanating from the center of projection (COP) passing through each point of the object and intersecting the projection plane.

Classification of Geometric Projections



Perspective Projections

Distance from COP to projection plane is finite. The projectors are not parallel & we specify a center of projection (COP).

Center of Projection is also called the Perspective Reference Point

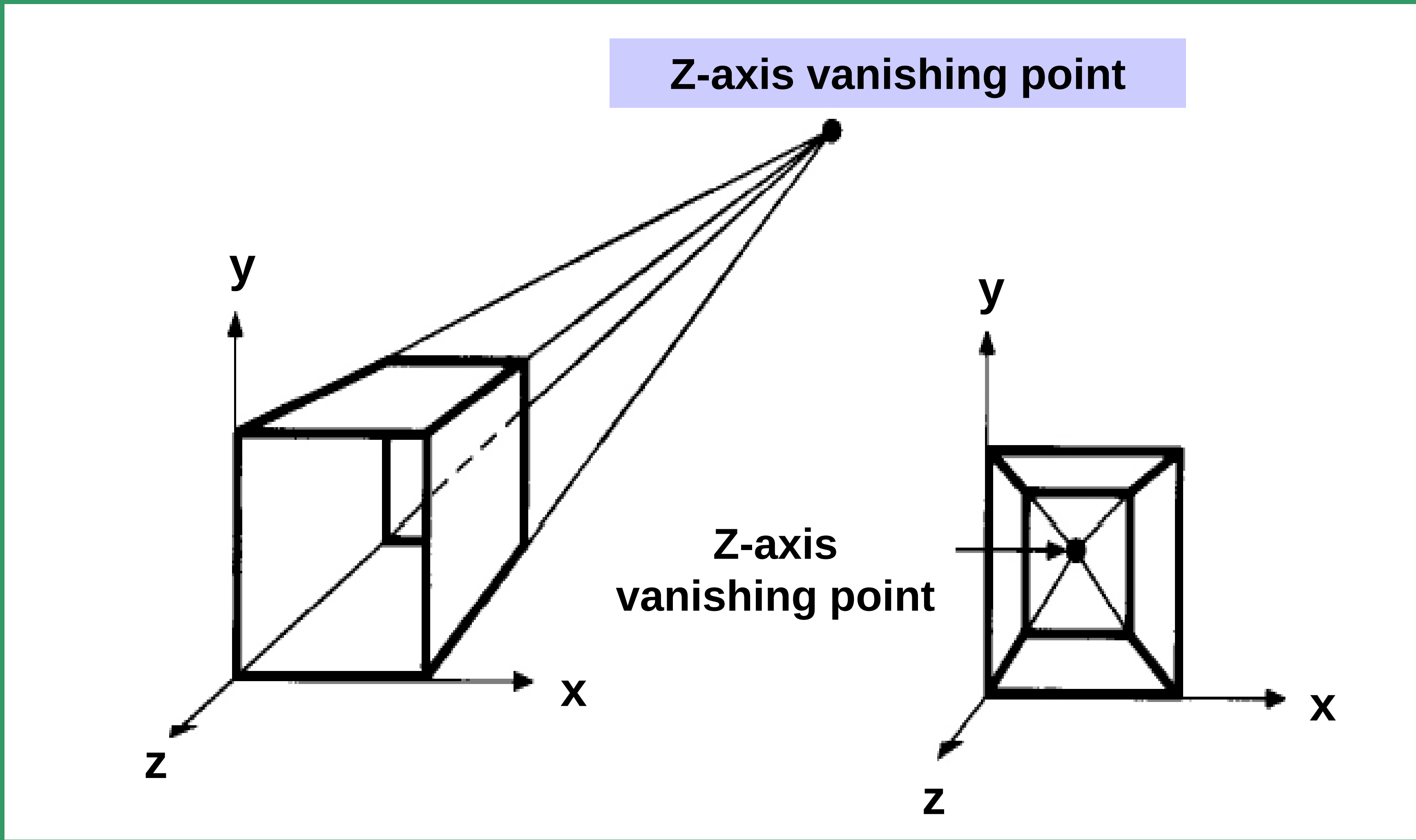
COP = PRP

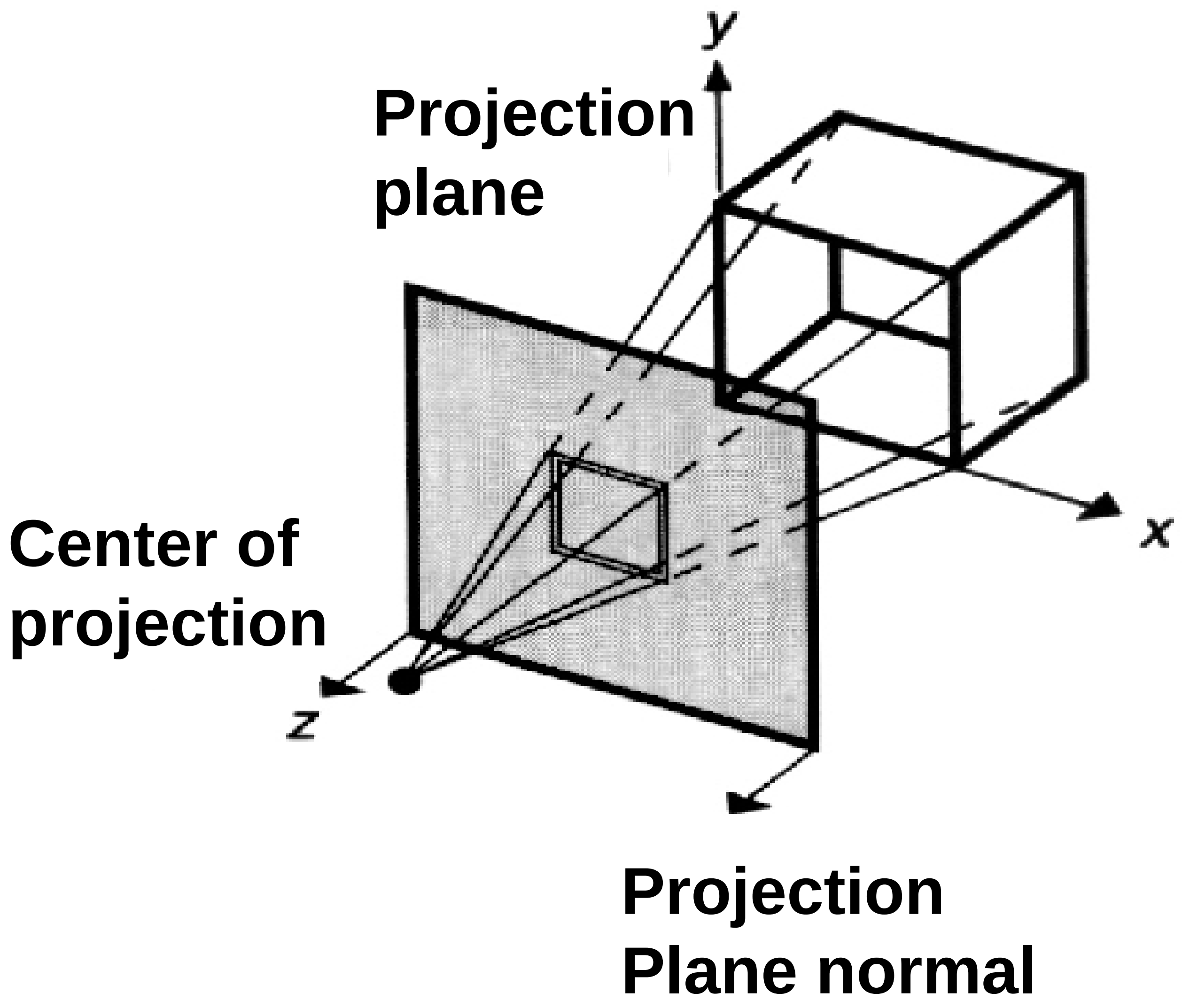
Perspective foreshortening:

The size of the perspective projection of the object varies inversely with the distance of the object from the center of projection.

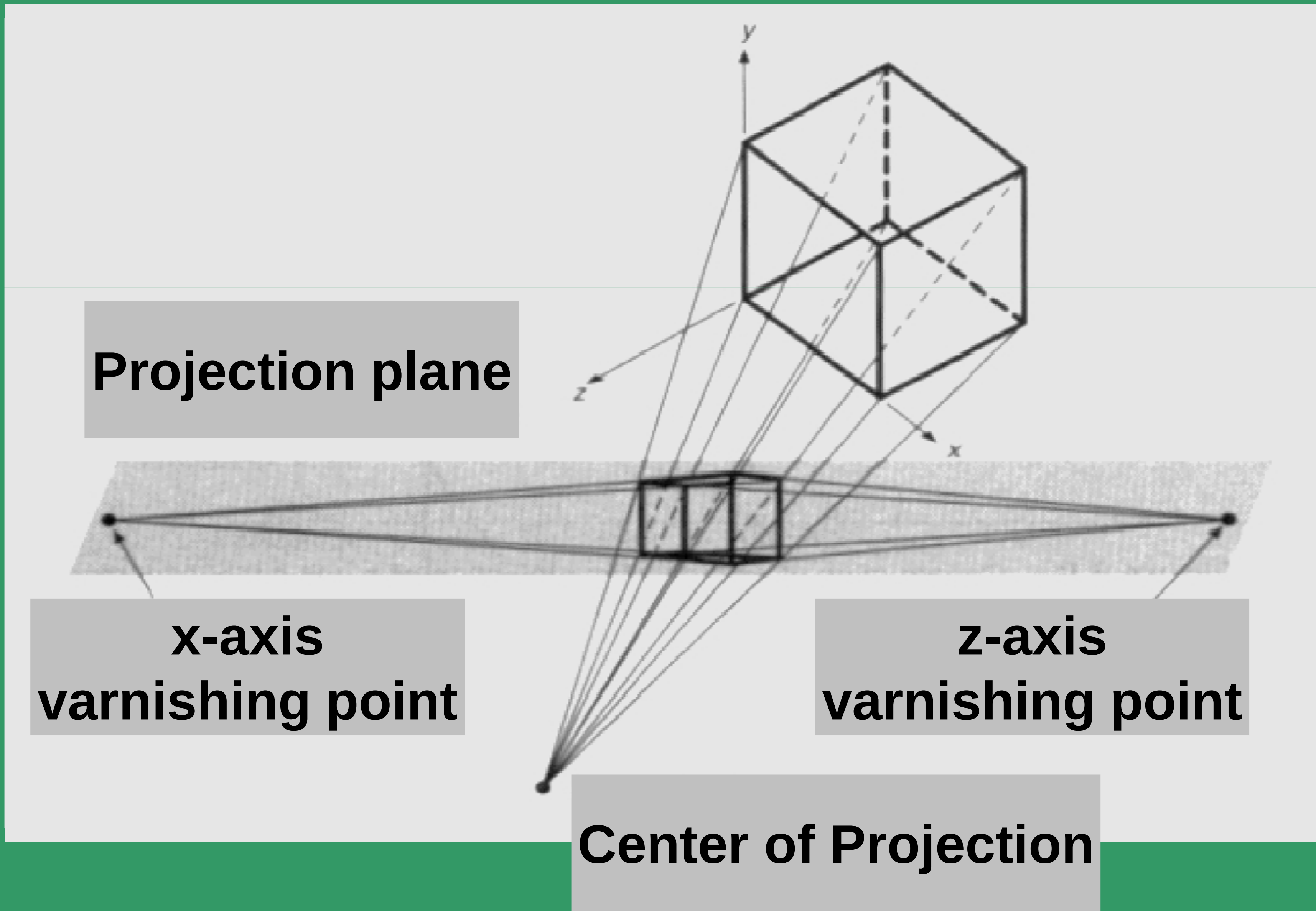
Vanishing Point:

The perspective projections of any set of parallel lines that are not parallel to the projection plane converge to a vanishing point.

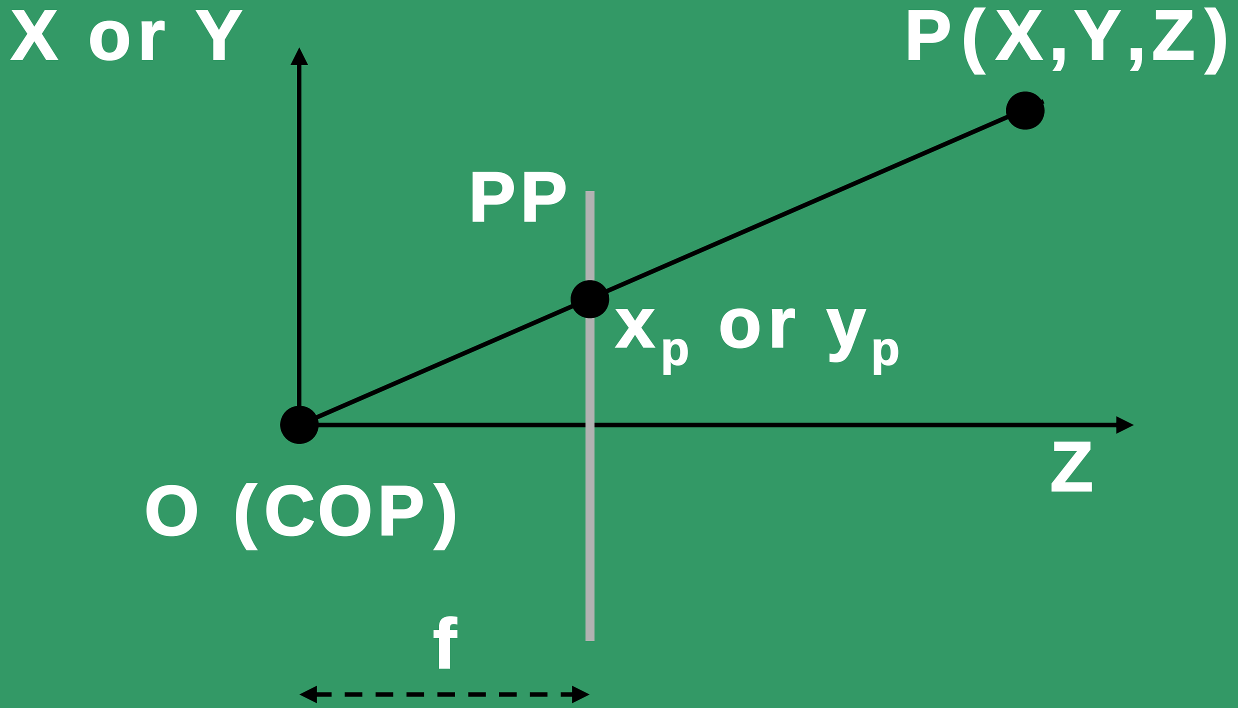

Z-axis vanishing point
y
x
z
y
Z-axis
vanishing point
x
z



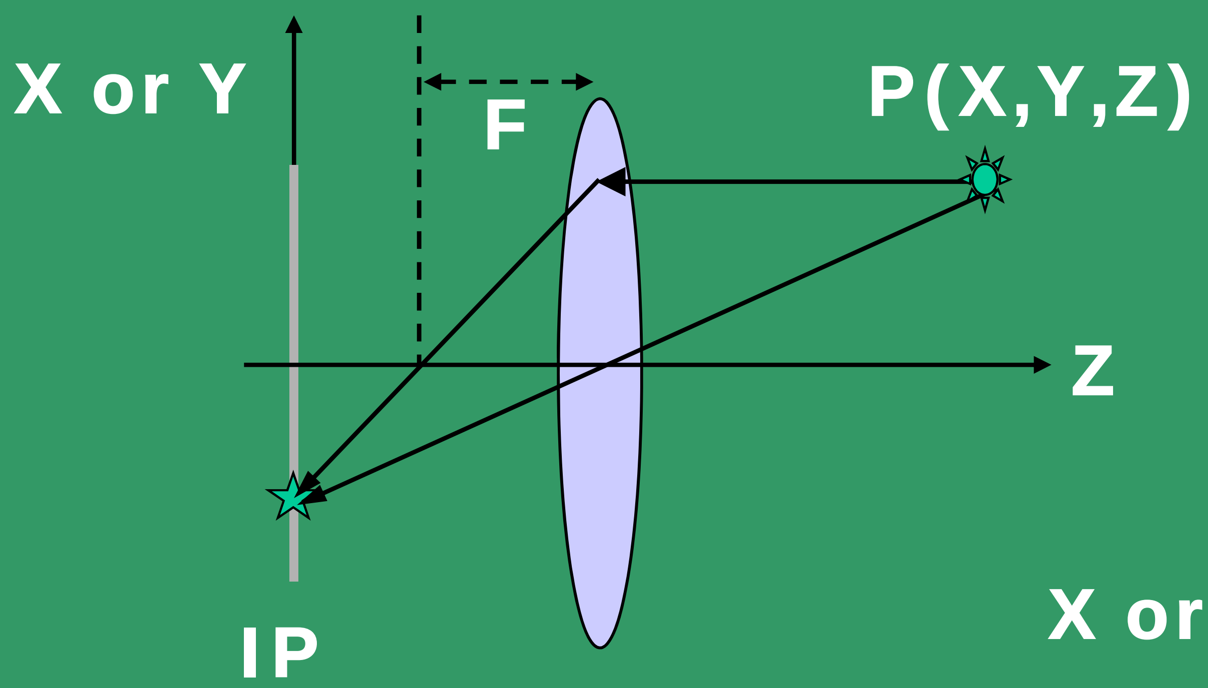
y
Projection plane
x
Center of projection
z
Projection Plane normal

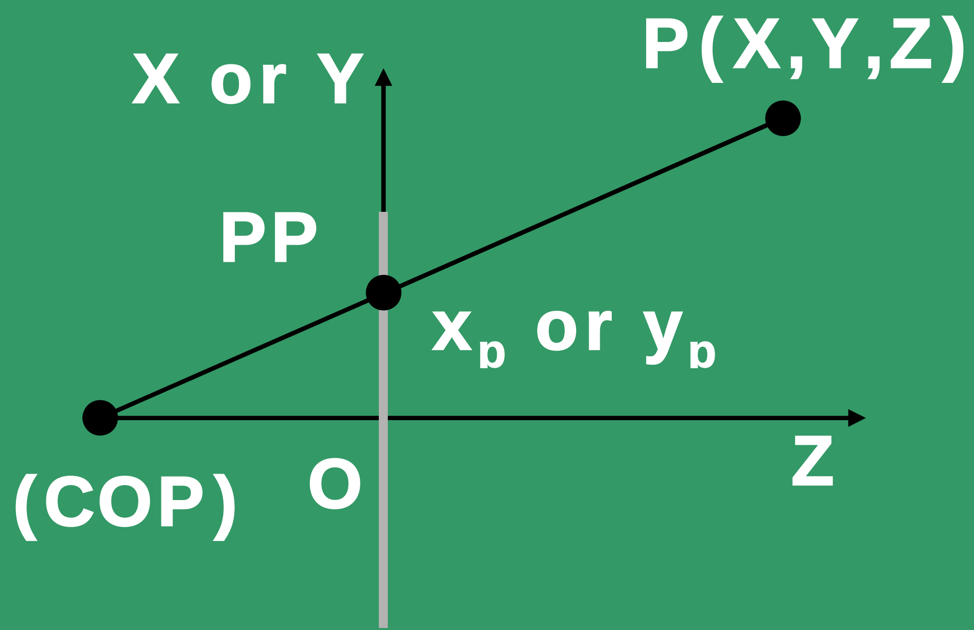

y
z
x
Projection plane
x-axis
varnishing point
z-axis
varnishing point
Center of Projection

Perspective Geometry and Camera Models

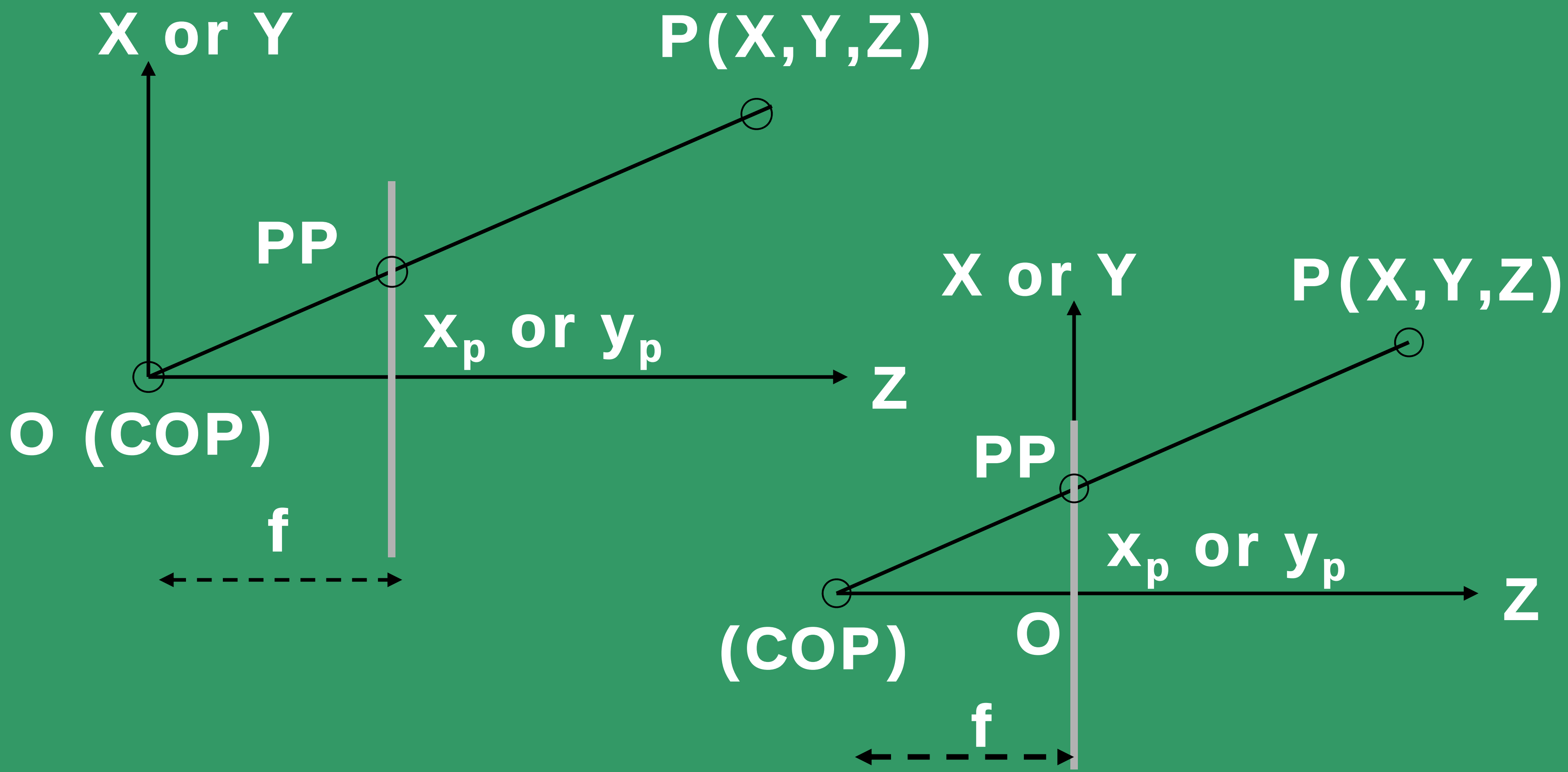


Perspective Geometry and Camera Models





Perspective Geometry and Camera Models



Equations of Perspective geometry, next ->

Equations of Perspective geometry

$$x_p/f = X/Z; \quad y_p/f = Y/Z;$$

$$M_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix}$$

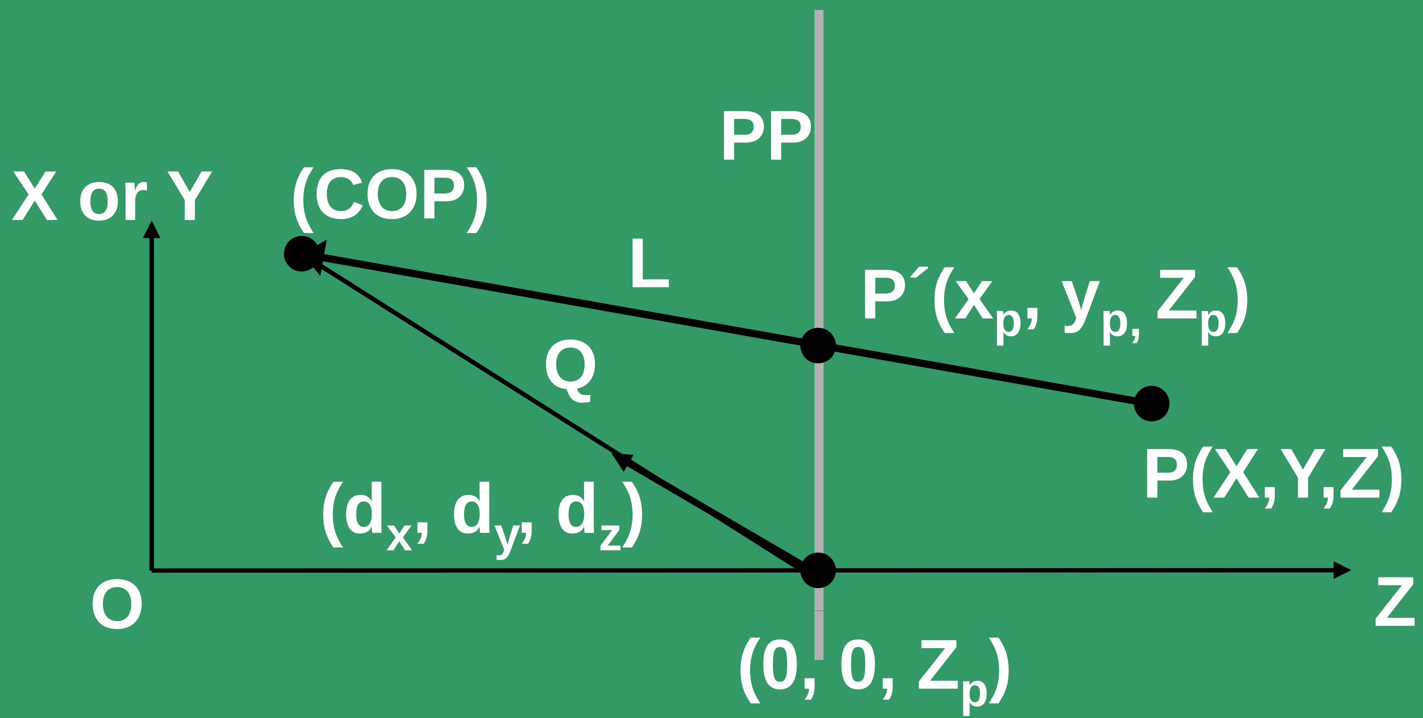
$P' = M_{per}.P;$

where, $P = [X\ Y\ Z\ 1]^T$

$$\frac{x_p}{f} = \frac{X}{Z+f}; \quad \frac{y_p}{f} = \frac{Y}{Z+f};$$

$$M_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 1 \end{bmatrix}$$

Generalized formulation of perspective projection:



Parametric eqn. of the line L between COP and P:

$$COP + t(P-COP); \quad 0 \leq t \leq 1.$$

Let the direction vector from $(0, 0, Z_p)$ to COP be (d_x, d_y, d_z),
and Q be the distance from $(0, 0, Z_p)$ to COP.

Then COP $= (0, 0, Z_p) + Q(d_x, d_y, d_z)$.

The coordinates of any point on line L is:

$$X' = Qd_x + (X - Qd_x)t;$$

$$Y' = Qd_y + (Y - Qd_y)t;$$

$$Z' = (Z_p + Qd_z) + (Z - (Z_p + Qd_z))t;$$

Using the condition $Z' = Z_p$, at the intersection of line L and plane PP:

$$t = \frac{-Qd_z}{Z - (Z_p + Qd_z)}$$

Now subsitute to obtain, x_p and y_p.

$$x_p = \frac{X - Z\dfrac{d_x}{d_z} + Z_p\dfrac{d_x}{d_z}}{\dfrac{Z_p - Z}{Qd_z} + 1}$$

$$y_p = \frac{Y - Z\dfrac{d_y}{d_z} + Z_p\dfrac{d_y}{d_z}}{\dfrac{Z_p - Z}{Qd_z} + 1}$$

Generalized formula of perspective projection matrix:

$$\mathbf{M}_{gen} = \begin{bmatrix} 1 & 0 & -\dfrac{d_x}{d_z} & z_p \dfrac{d_x}{d_z} \\ 0 & 1 & -\dfrac{d_y}{d_z} & z_p \dfrac{d_y}{d_z} \\ 0 & 0 & -\dfrac{Z_p}{Qd_z} & \dfrac{Z_p^2}{Qd_z} + Z_p \\ 0 & 0 & -\dfrac{1}{Qd_z} & \dfrac{Z_p}{Qd_z} + 1 \end{bmatrix}$$

Special cases from the generalized formulation of the perspective projection matrix

Matrix Type	Z_p	Q	$[d_x, d_y, d_z]$
M_{orth}	0	Infinity	[0, 0, -1]
M_{per}	d	d	[0, 0, -1]
M'_{per}	0	d	[0, 0, -1]

If Q is finite, M_{gen} defines a one-point perspective projection in the above two cases.

Parallel Projection

Distance from COP to projection plane is infinite.

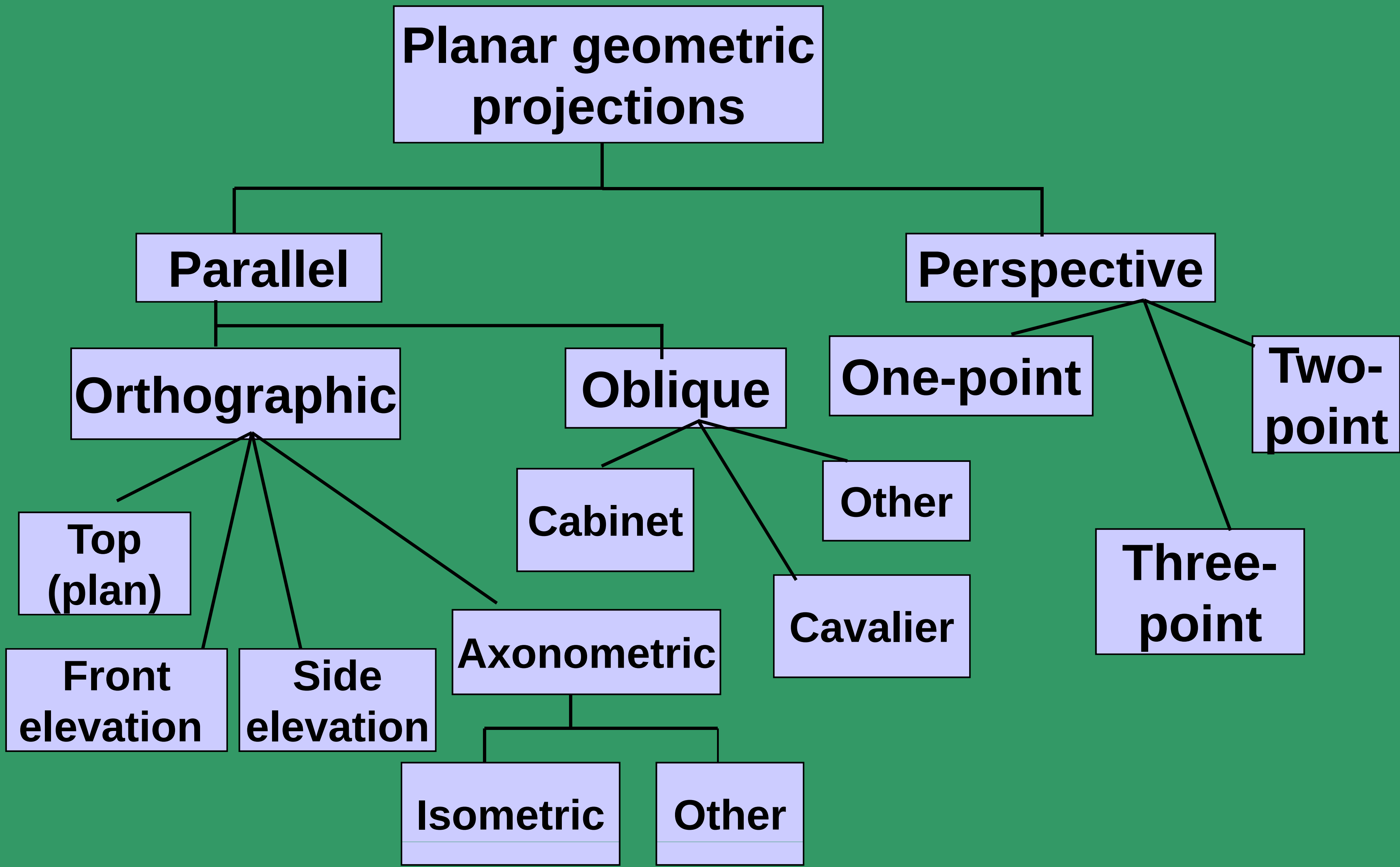
Therefore, the projectors are parallel lines & we need to specify a:

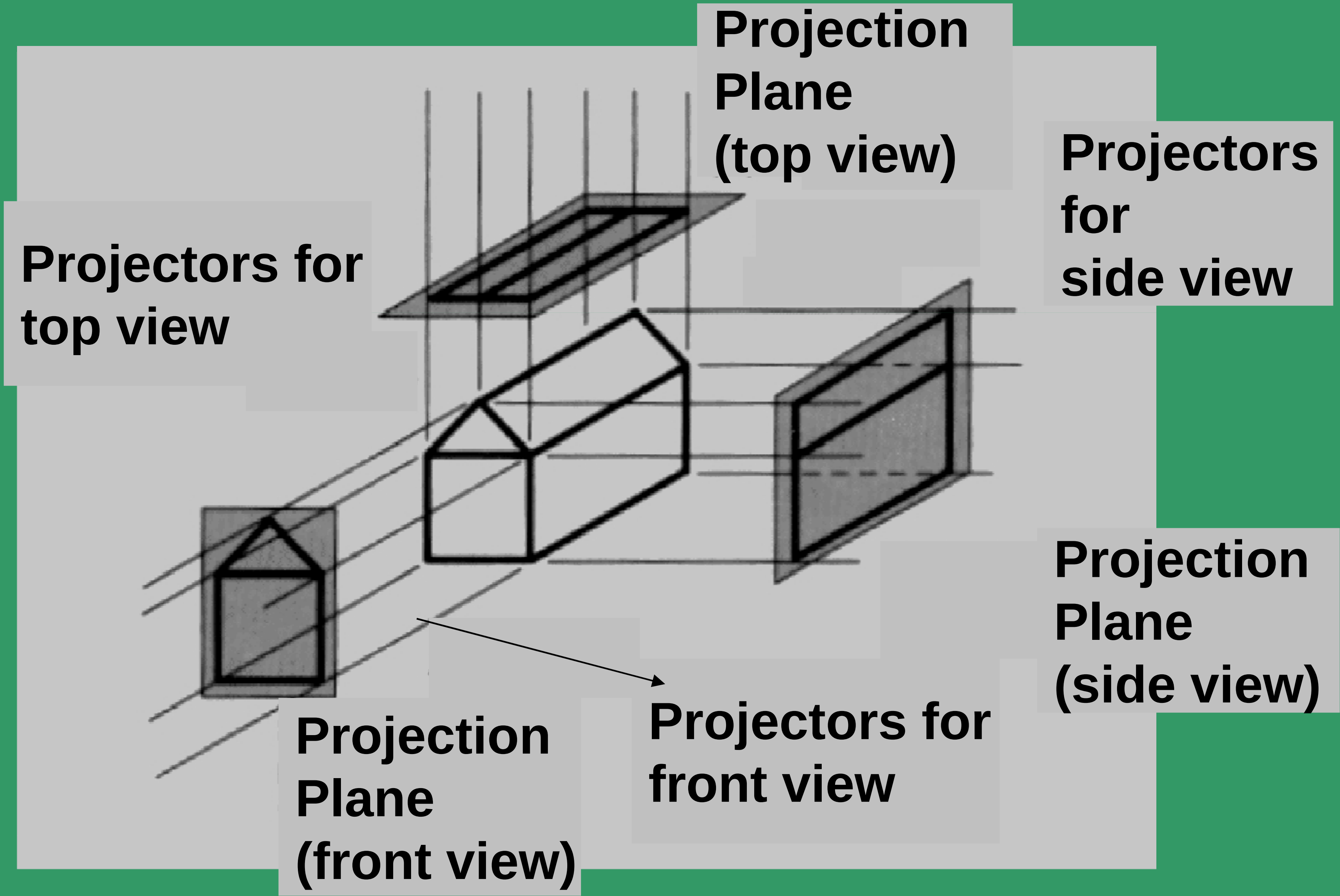
direction of projection (DOP)

Orthographic:

the direction of projection and the normal to the projection plane are the same. (direction of projection is normal to the projection plane).

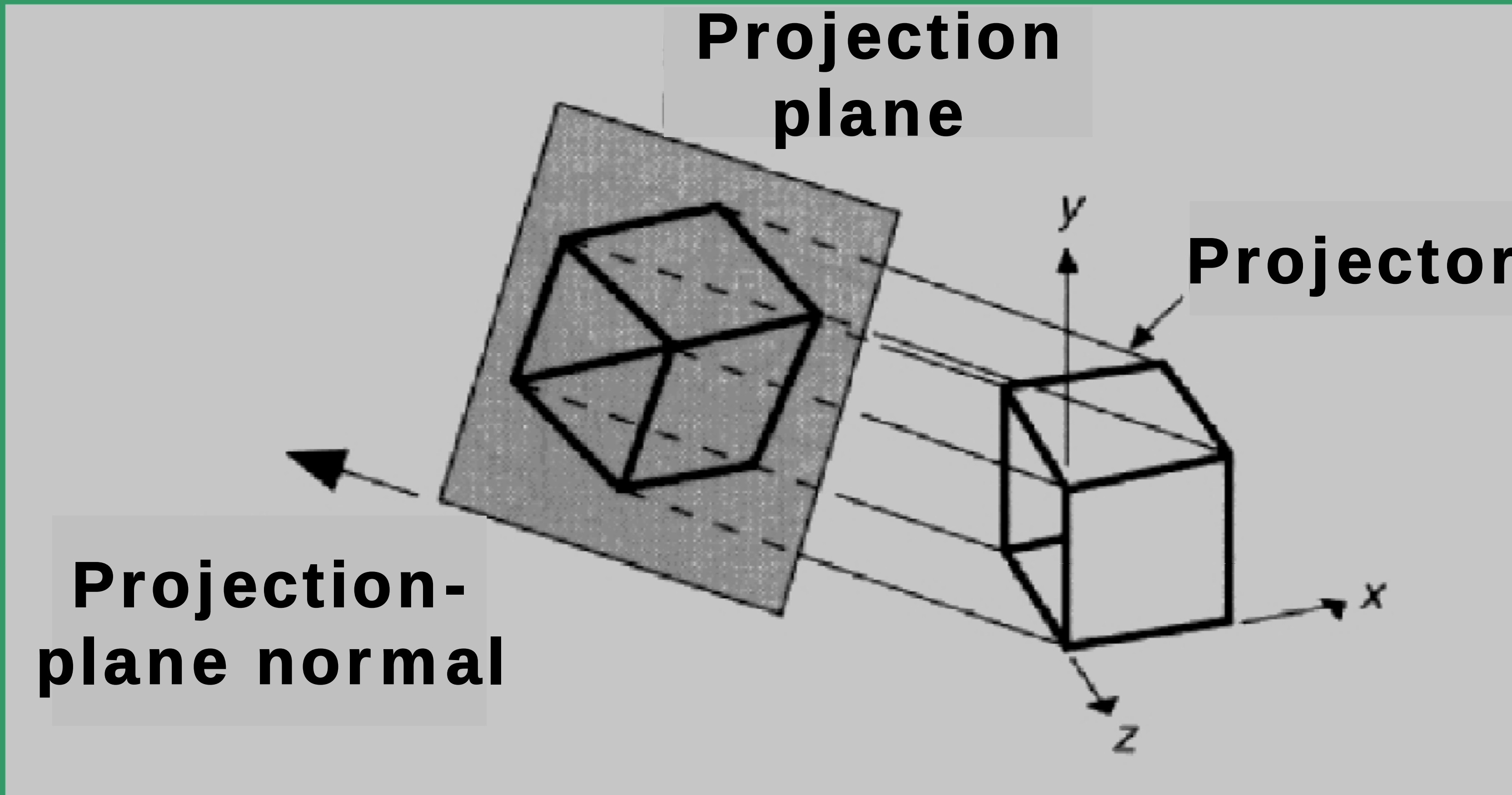
Classification of Geometric Projections





Example of Orthographic Projection

Example of Isometric Projection:



Axonometric orthographic projections use planes of projection that are not normal to a principal axis (they therefore show multiple face of an object.)

Isometric projection: projection plane normal makes equal angles with each principle axis. DOP Vector: [1 1 1].

All 3 axis are equally foreshortened allowing measurements along the axes to be made with the same scale.

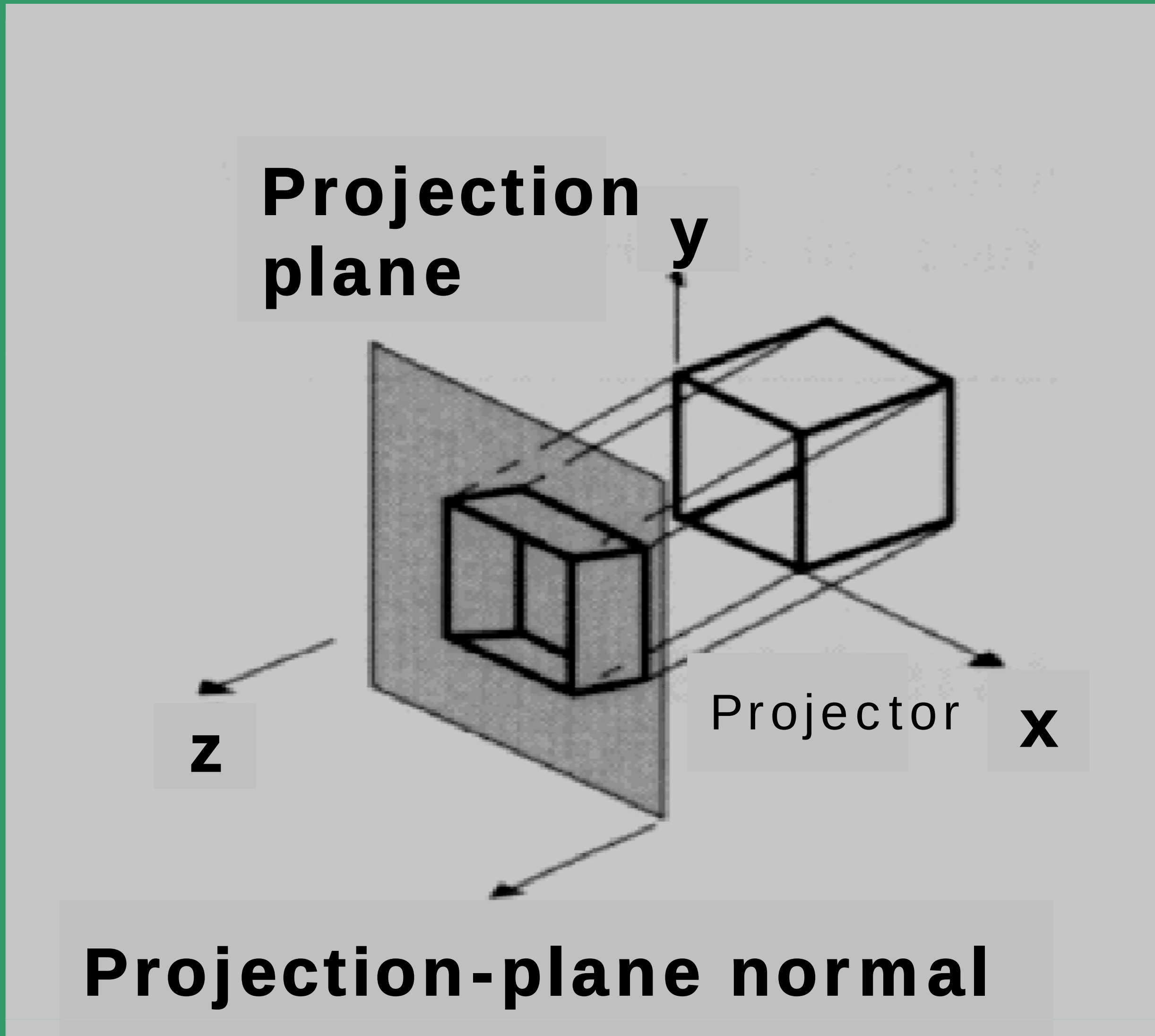
Oblique projections :

projection plane normal and the direction of projection differ.

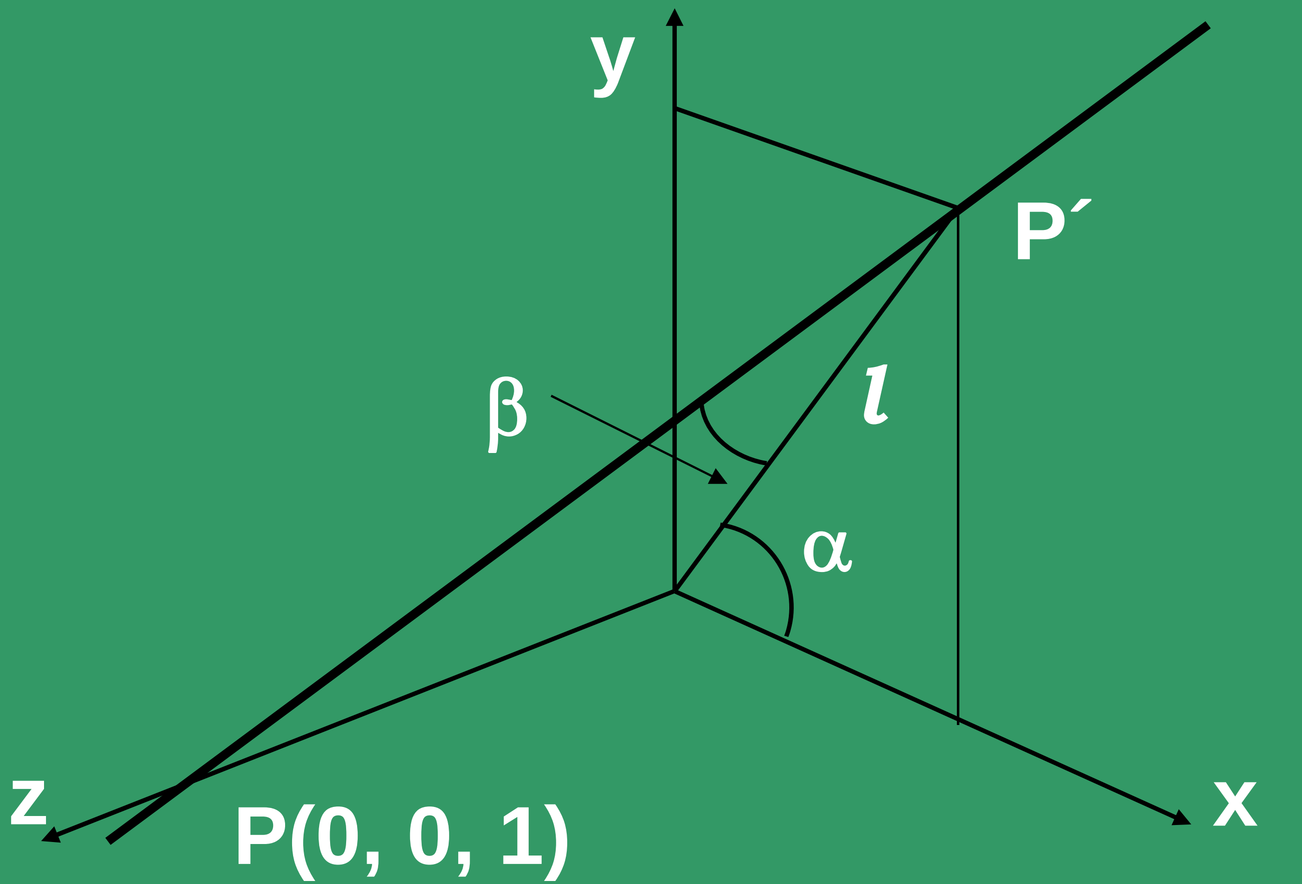
Plane of projection is normal to a Principle axis

Projectors are not normal to the projection plane

Example Oblique Projection



General oblique projection of a point/line:



General oblique projection of a point/line:

Projection Plane: x-y plane; P´ is the projection of P(0, 0, 1) onto x-y plane.

`l´ is the projection of the z-axis unit vector onto x-y plane and α is the angle the projection makes with the x-axis.

When DOP varies, both `l´ and α will vary.

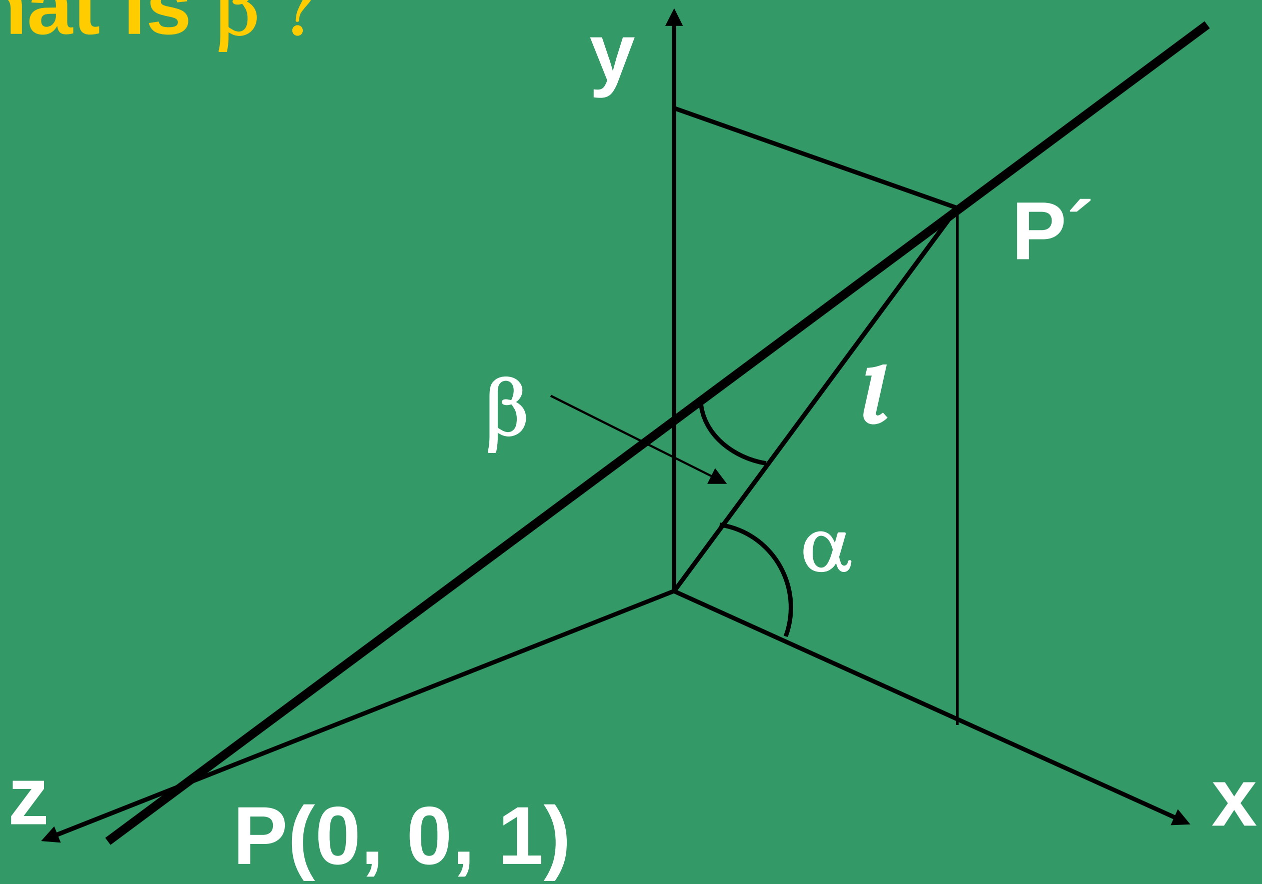
Coordinates of P´: (l *cos* α, l *sin* α, 0).

As given in the figure: DOP is:

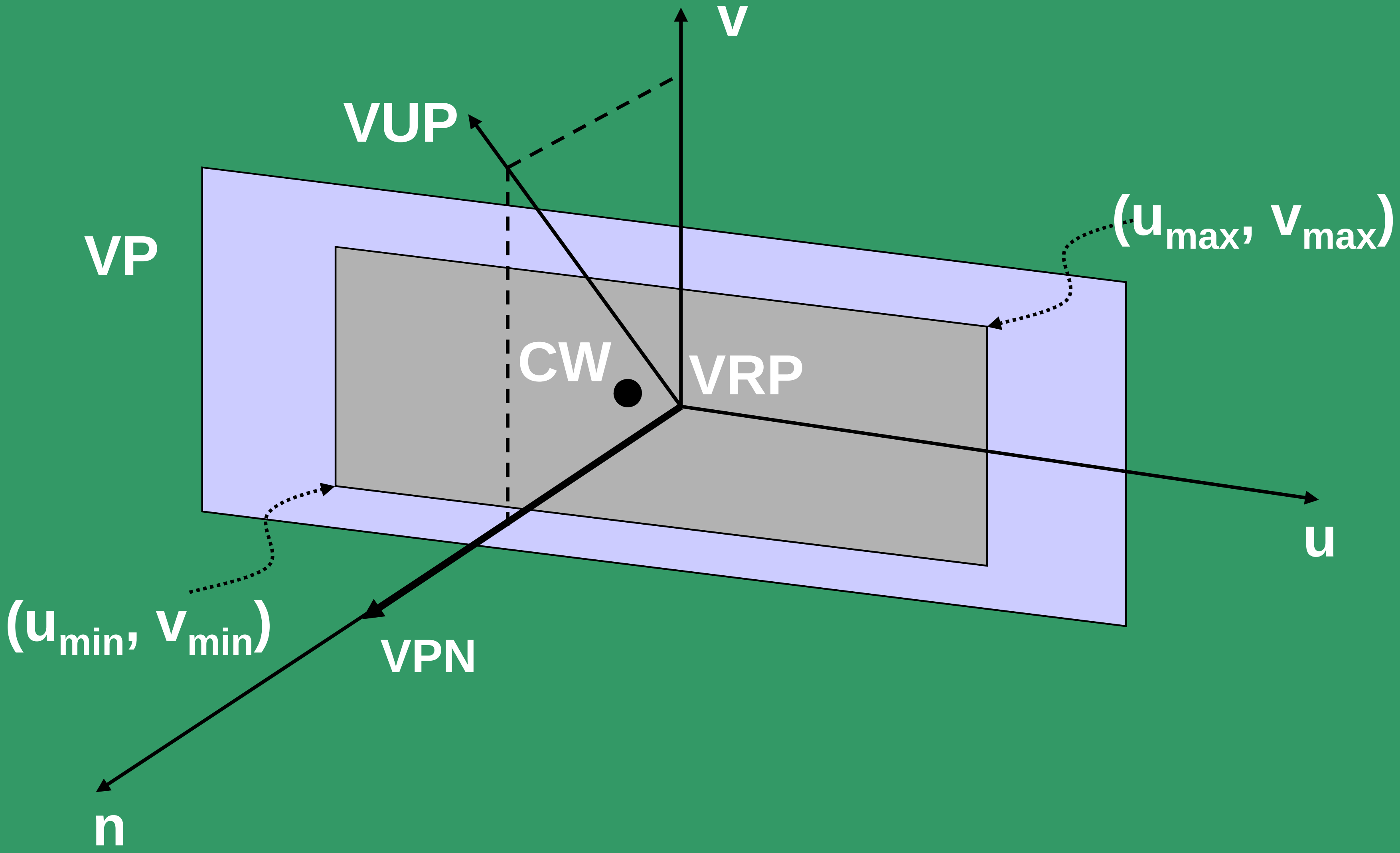
(d_x, d_y, -1) or (l *cos* α, l *sin* α, -1).

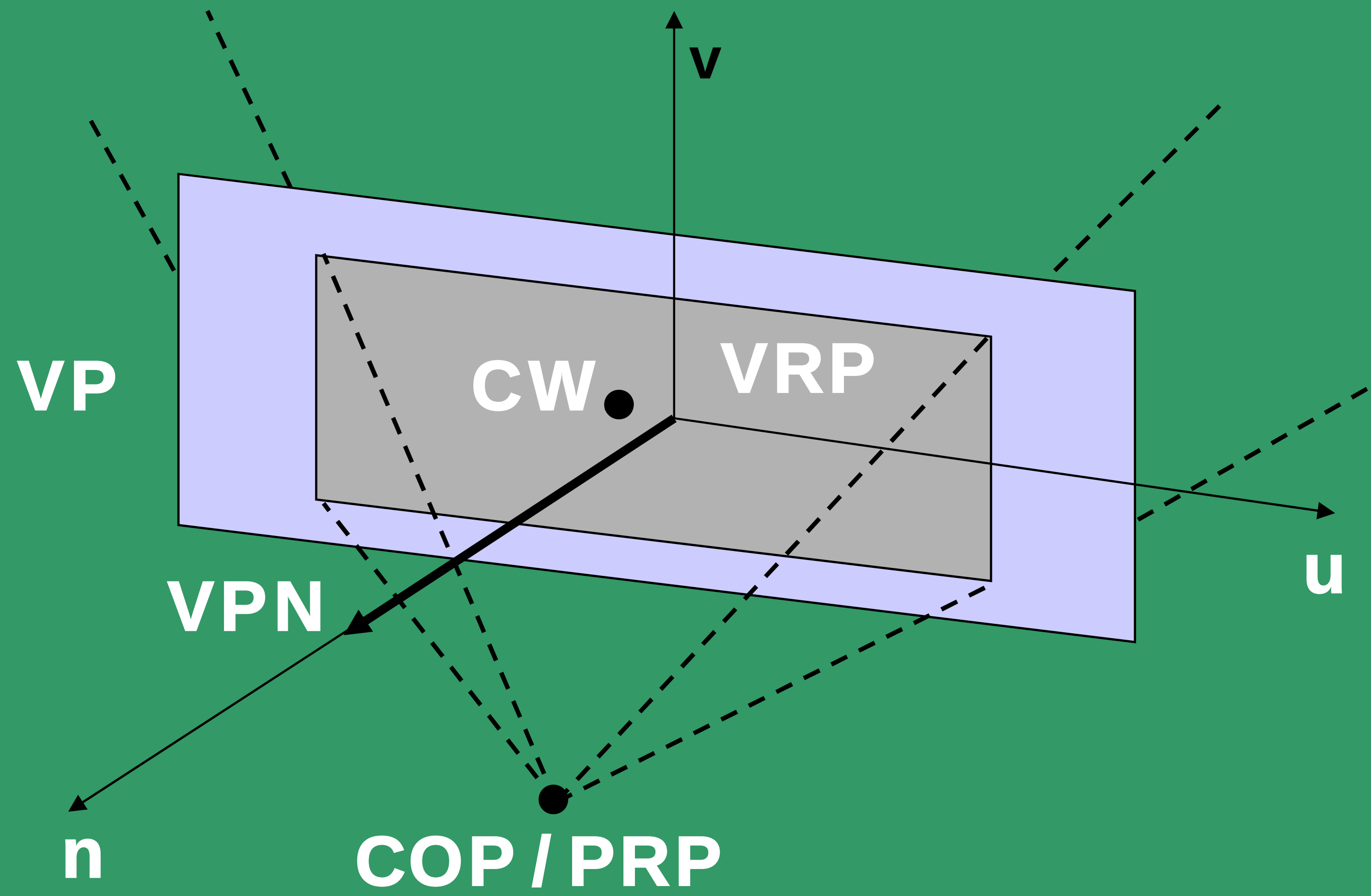
General oblique projection of a point/line:

What is β ?

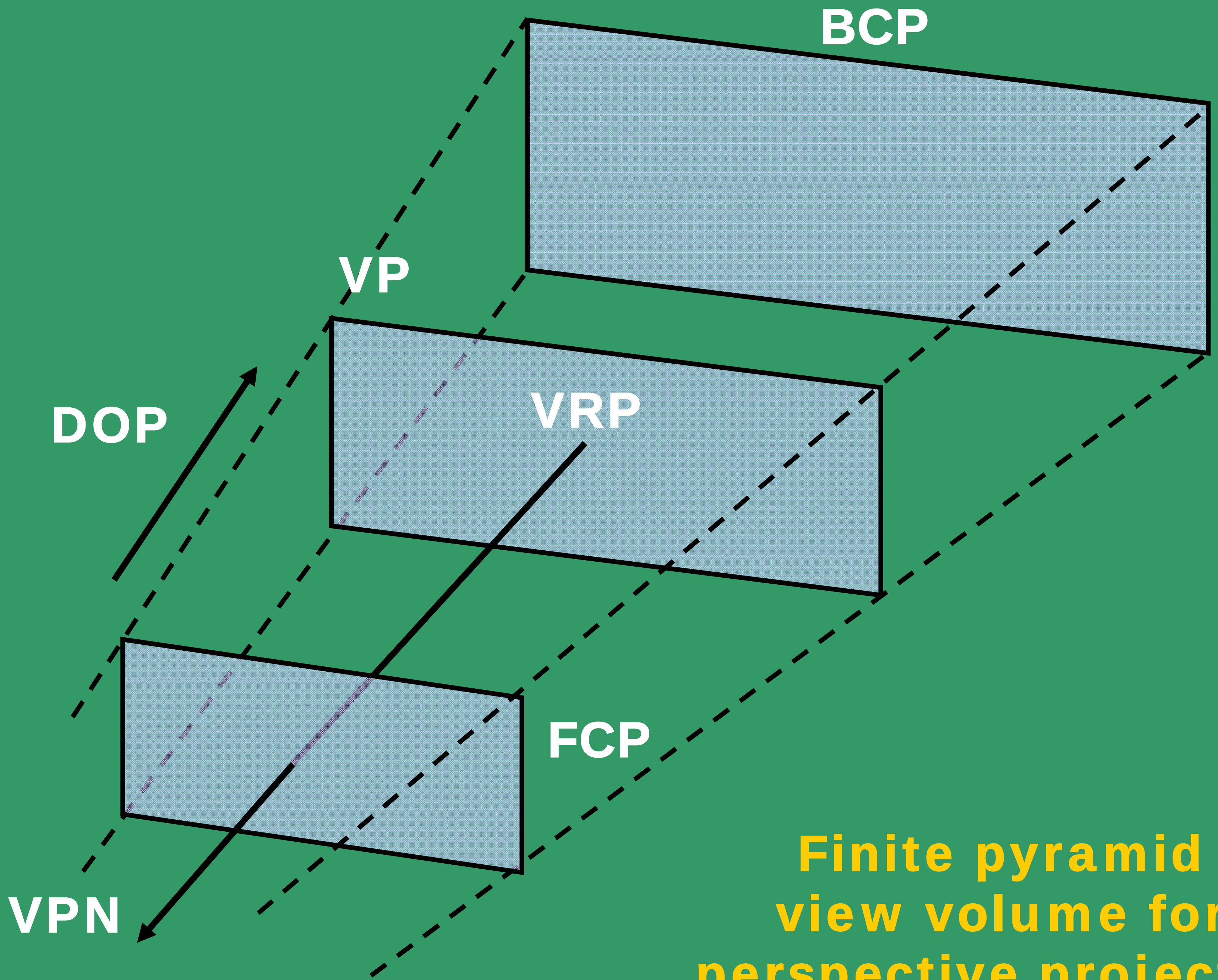


View Specifications:
VP, VRP, VUP, VPN, PRP, DOP, CW, VRC

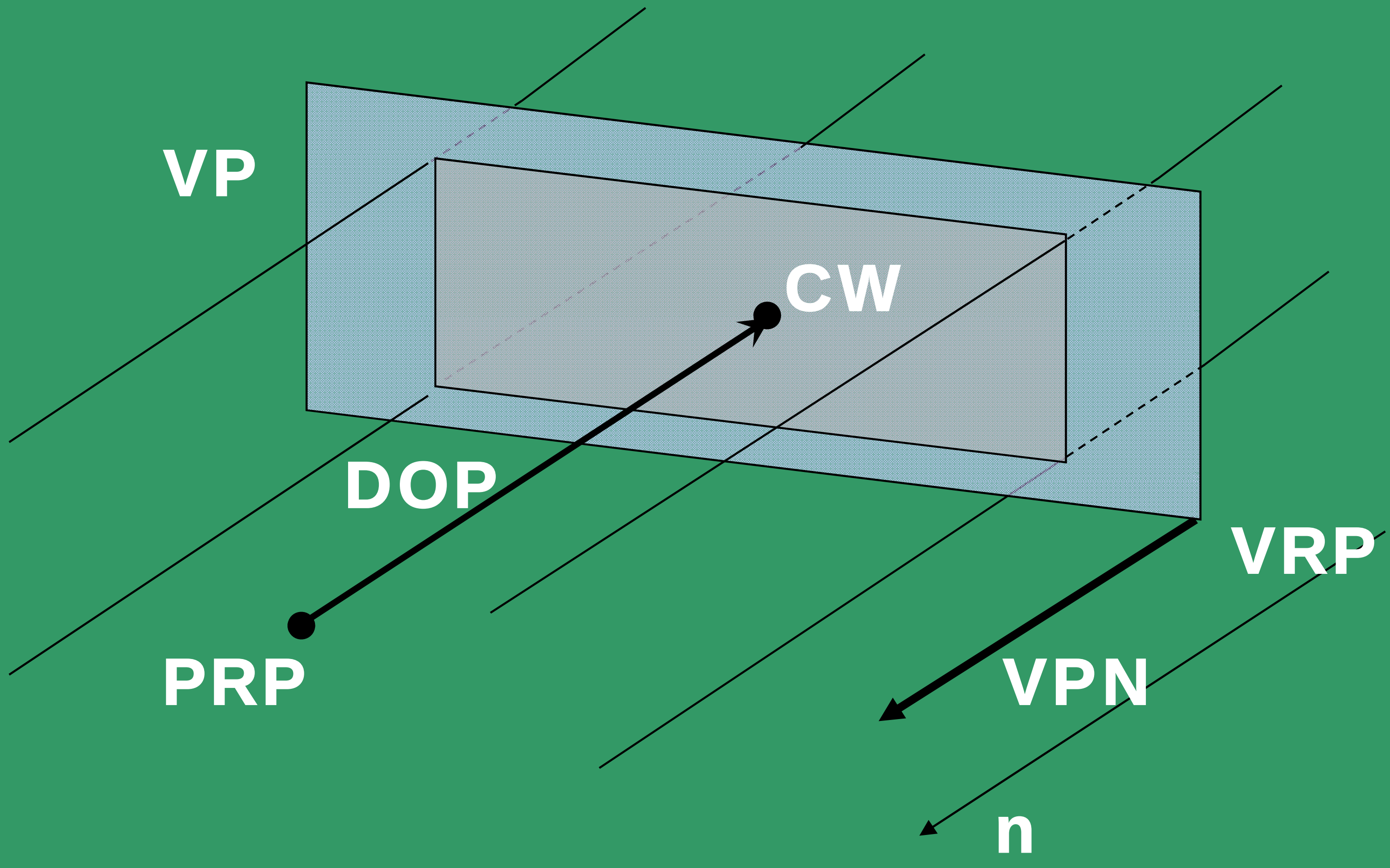




Semi-infinite pyramid view volume
for perspective projection



Finite pyramid view volume for perspective projection



Infinite parallelopiped view volume for parallel projection

Finite parallelopiped view volume for parallel projection