Linear Algebra and Random Processes

LARP/2018
CS6015

Course exclusively for CSE-PG students;
Any UG or non-CSE PG may meet HOD-CSE, for their requirements

Linear Algebra:

- **Matrices**
  - **Matrix Multiplication, Transposes, Inverses, Gaussian Elimination, Matrix norms, factorization A=LU, rank.**

- **Vector spaces**
  - **Column and row spaces, Solving Ax=0 and Ax=b, Independence, basis, dimension, linear transformations.**

- **Orthogonality**
  - **Orthogonal vectors and subspaces, Normal matrix, projection and least squares, Gram-Schmidt orthogonalization, QR factorization, Cholesky decomposition, polar decomposition.**

- **Determinants**
  - **Determinant formula, cofactors, inverses and volume.**

- **Eigenvalues and Eigenvectors**
  - **Characteristic polynomial, Diagonalization, Hermitian and Unitary matrices, Toeplitz matrices, Spectral theorem, Change of basis.**

- **Positive definite matrices**
  - **Positive definite and semi-definite matrices, Pseudo-inverse, Singular Value Decomposition (SVD), RRQR factorization.**

Random processes:

Preliminaries

- **Events, probability, conditional probability, independence, product spaces.**

Random Variables

- **Distributions, law of averages, discrete and continuous r.v.s, random vectors, Monte Carlo simulation.**

Discrete Random Variables

- **Probability mass functions, independence, expectation, conditional expectation, sums of r.v.s.**

Continuous Random Variables

- **Probability density functions, independence, expectation, conditional expectation, functions of r.v.s, sum of r.v.s, multivariate normal distribution, sampling from a distribution.**

Convergence of Random Variables

- **Modes of convergence, Borel-Cantelli lemmas, laws of large numbers, central limit theorem, tail inequalities.**

Estimation of parameters from data

- **Kullback-Leibler Divergence, Principal Component Analysis, Chi-square test, Student's T-test, Maximum Likelihood Estimate, Expectation-Maximization.**

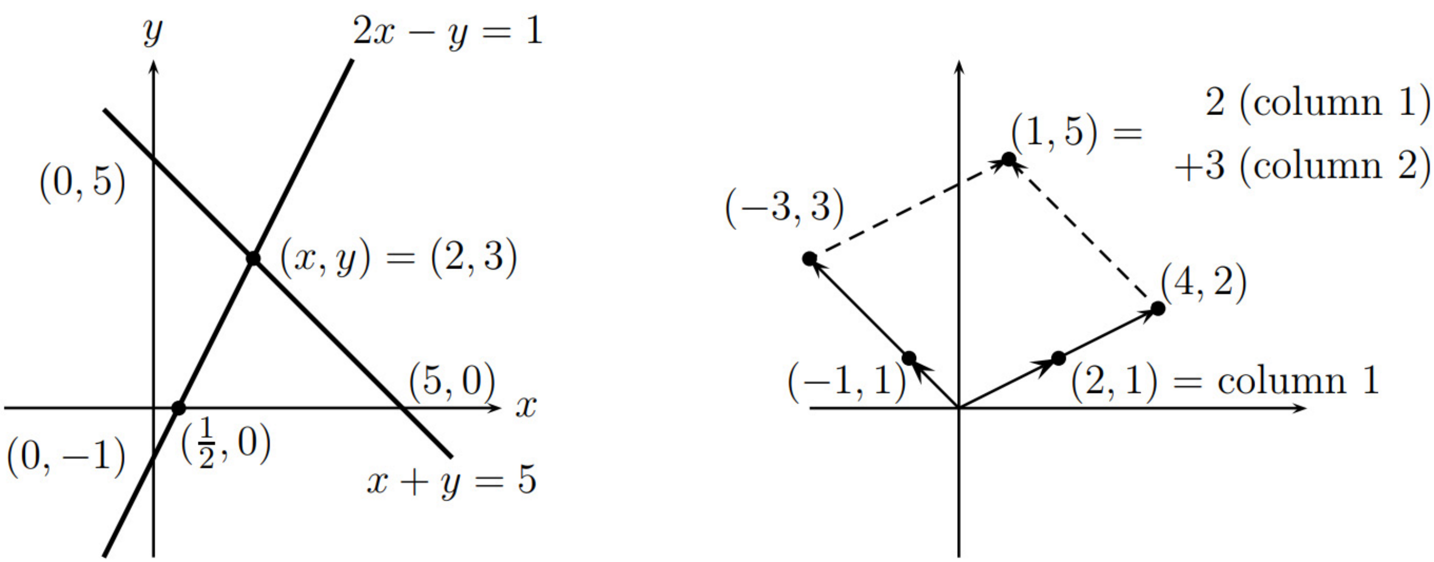
Advanced topics

- **Markov chains, minimum mean squared error estimation.**

Geometry of Linear Equations (2-D space)

2 lines : $\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$ $\qquad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Soln ??



(a) Lines meet at $x = 2$, $y = 3$ (b) Columns combine with 2 and 3

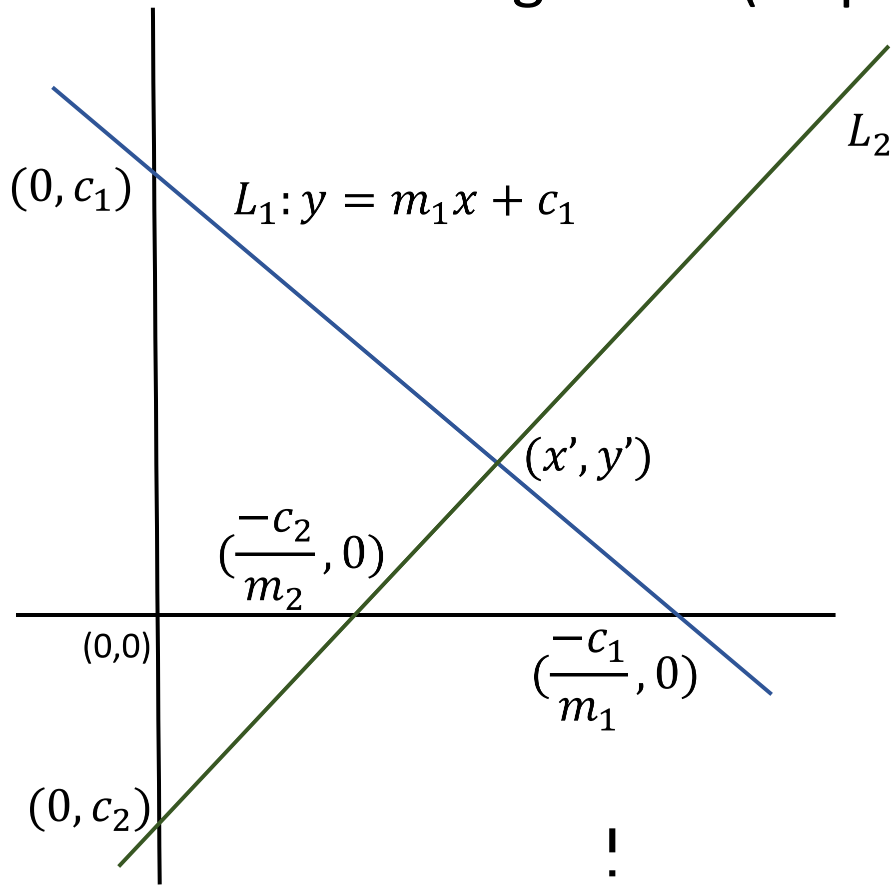
Row picture (two lines) and column picture (combine columns).

Two separate equations are really **one vector equation**

$$\text{Column form : } x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

The problem is to **find the combination of the column vectors on the left side that produces the vector on the right side**.

$y = mx + c$ form of straight line (slope-intercept form)



$L_1: \; y - m_1x = c_1$
$L_2: \; y - m_2x = c_2$

m_1 : slope of L_1
m_2 : slope of L_2

(x', y'): point of intersection ??

$$x' = \frac{\boldsymbol{c_2 - c_1}}{\boldsymbol{m_1 - m_2}},$$

$$y' = \frac{\boldsymbol{c_2 m_1 - c_1 m_2}}{\boldsymbol{m_1 - m_2}}$$

$AX = B$

$$\begin{bmatrix} -m_1 & 1 \\ -m_2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

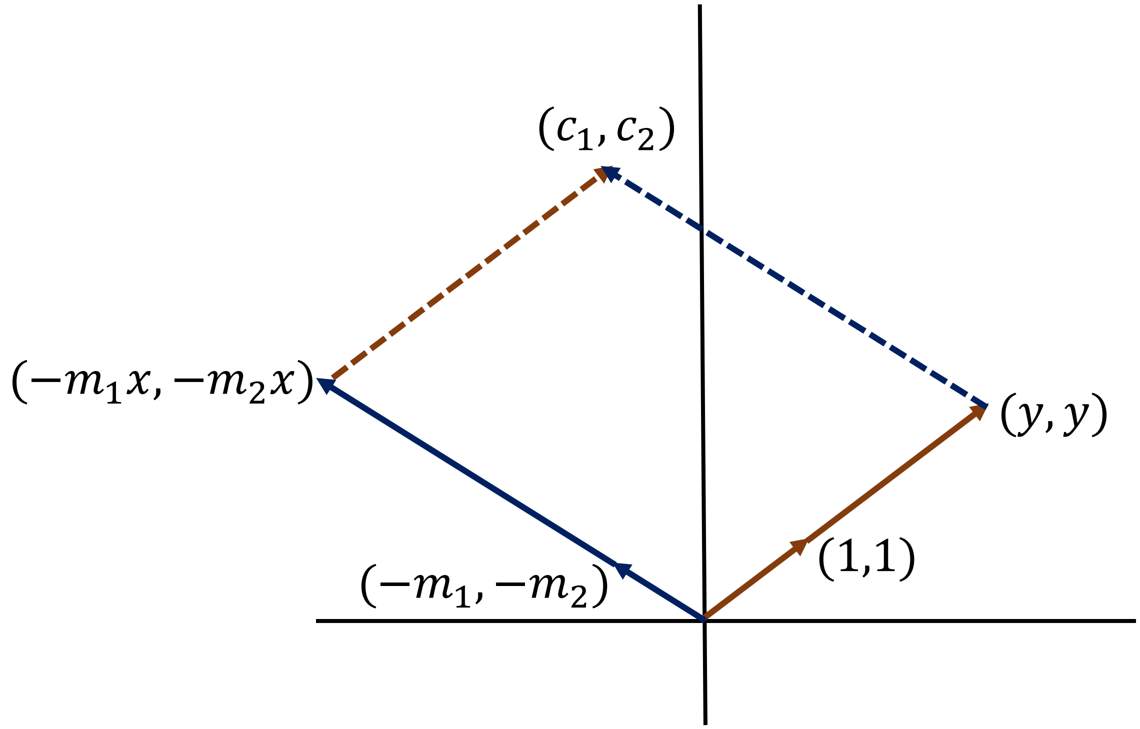
Augmented matrix

Co-efficient matrix

Column form : $x \begin{bmatrix} -m_1 \\ -m_2 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

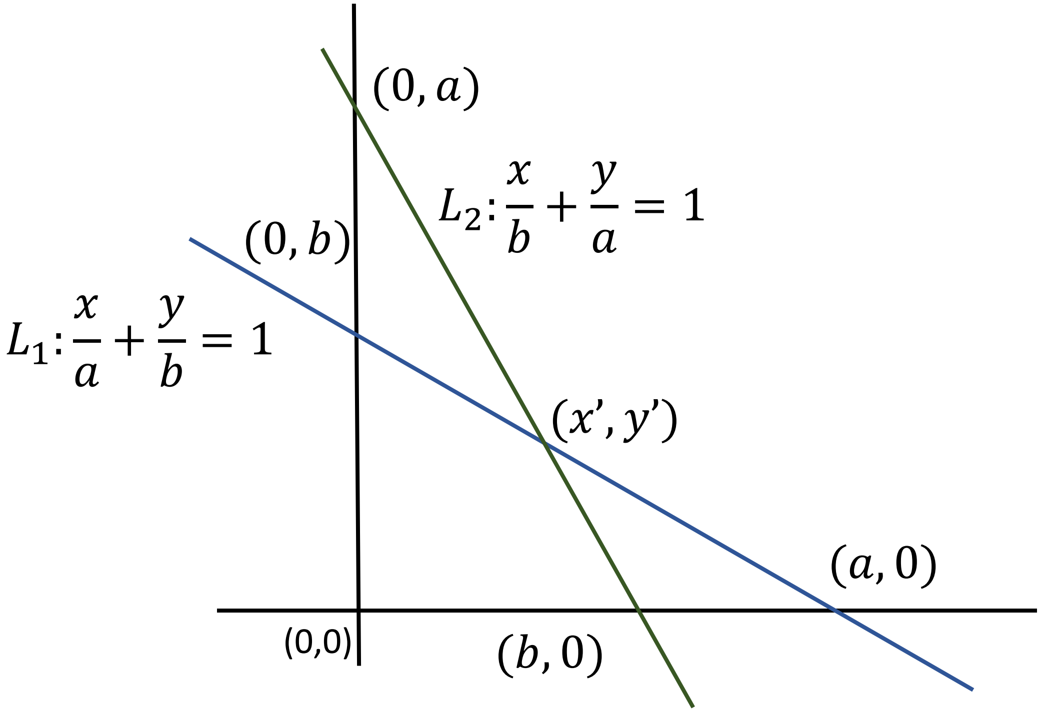
$y = mx + c$ form of straight line (slope-intercept form)

Column form : $x\begin{bmatrix}-m_1\\-m_2\end{bmatrix} + y\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$



x times column 1 combines with y times column 2
to produce column 3

$\frac{x}{a} + \frac{y}{b} = c$: form of straight line (intercept form)



$L_1: bx + ay = c_1$
$L_2: ax + by = c_2$

(x', y'):point of intersection

$$AX = B$$

$$\begin{bmatrix} b & a \\ a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Augmented matrix

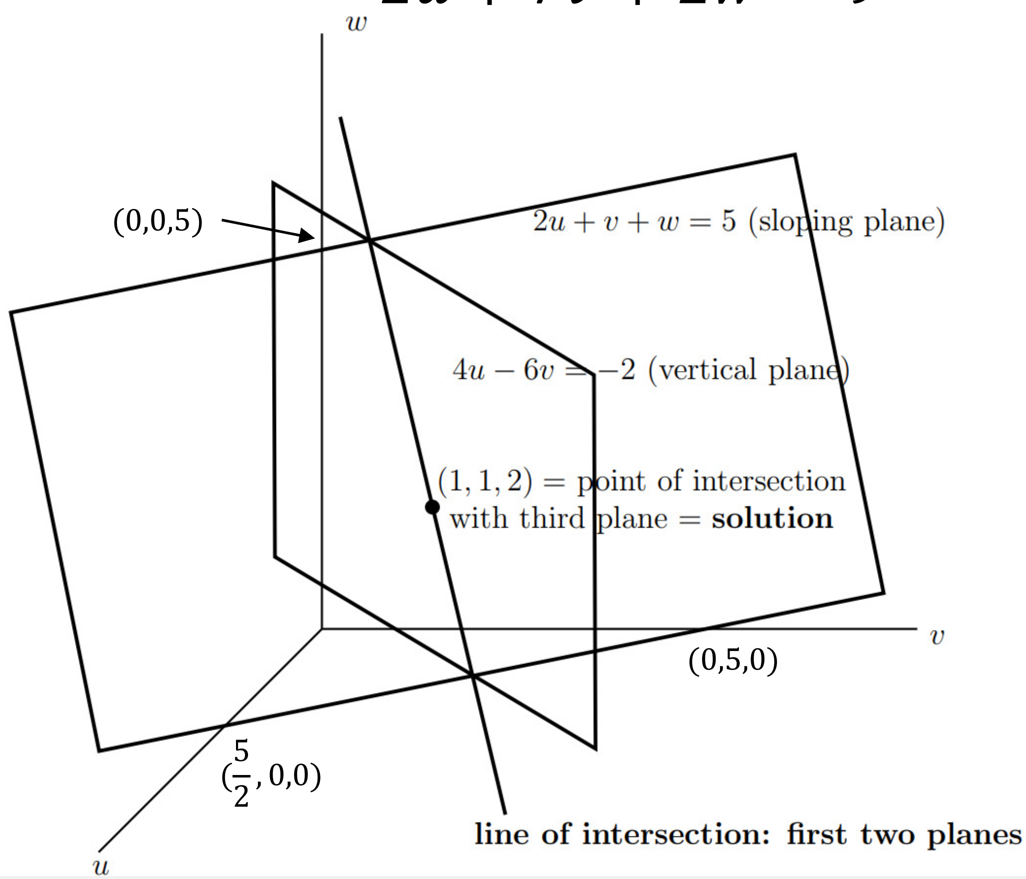
Co-efficient matrix

Column form : $x \begin{bmatrix} b \\ a \end{bmatrix} + y \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Geometry of Linear Equations (3-D space)

3 planes:

$$2u + v + w = 5$$
$$4u - 6v = -2$$
$$-2u + 7v + 2w = 9$$



The row picture: three intersecting planes from three linear equations.

3rd plane not shown for ease of understanding

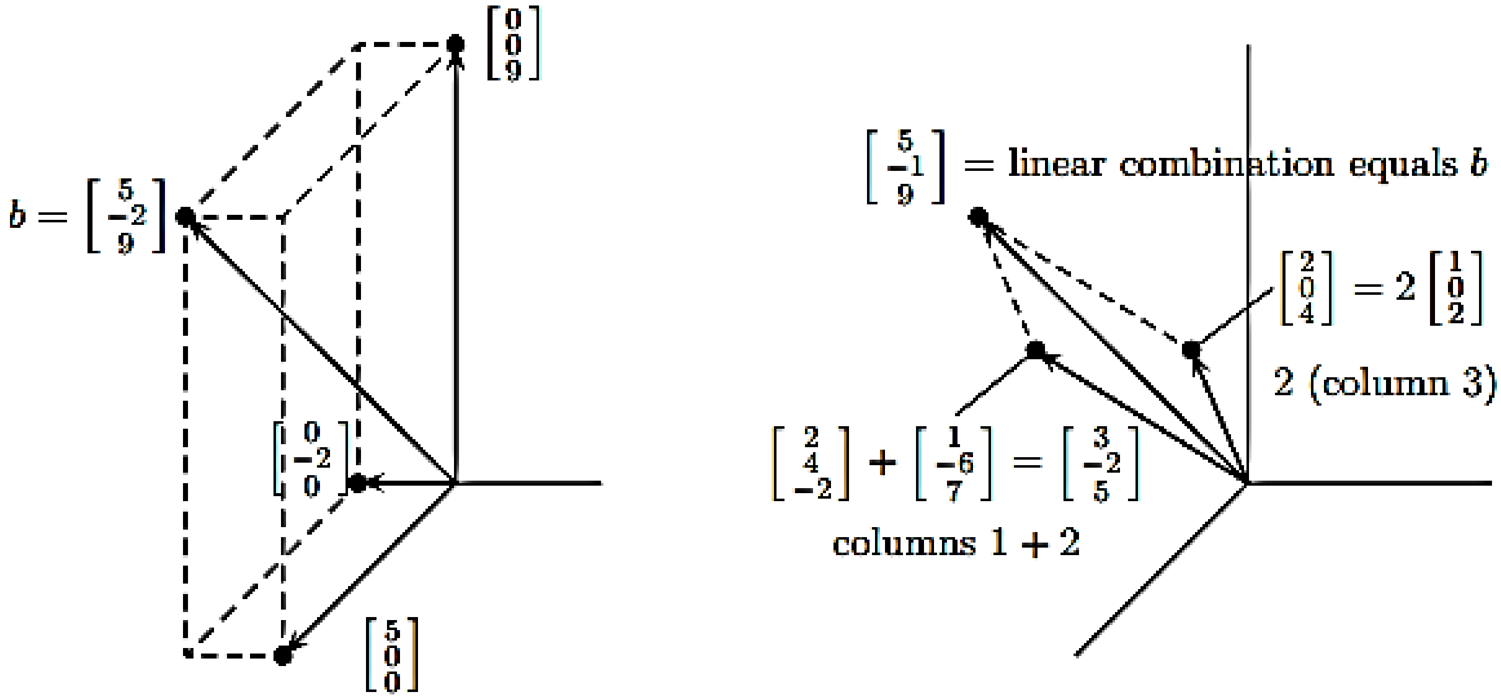
Geometry of Linear Equations (3-D space)

3 planes:

$$2u + v + w = 5$$
$$4\text{u} - 6v = -2$$
$$-2u + 7v + 2w = 9$$

Column form : $\text{u}\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v\begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

$$\text{Vector addition}: \begin{bmatrix}5\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-2\\0\end{bmatrix} + \begin{bmatrix}0\\0\\9\end{bmatrix} = \begin{bmatrix}5\\-2\\9\end{bmatrix}$$



(a) Add vectors along axes

(b) Add columns 1 + 2 + (3 + 3)

The COLUMN Picture

$$\text{Linear Combination}: 1\begin{bmatrix}2\\4\\2\end{bmatrix} + 1\begin{bmatrix}1\\-6\\7\end{bmatrix} + 2\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}5\\-2\\9\end{bmatrix}$$

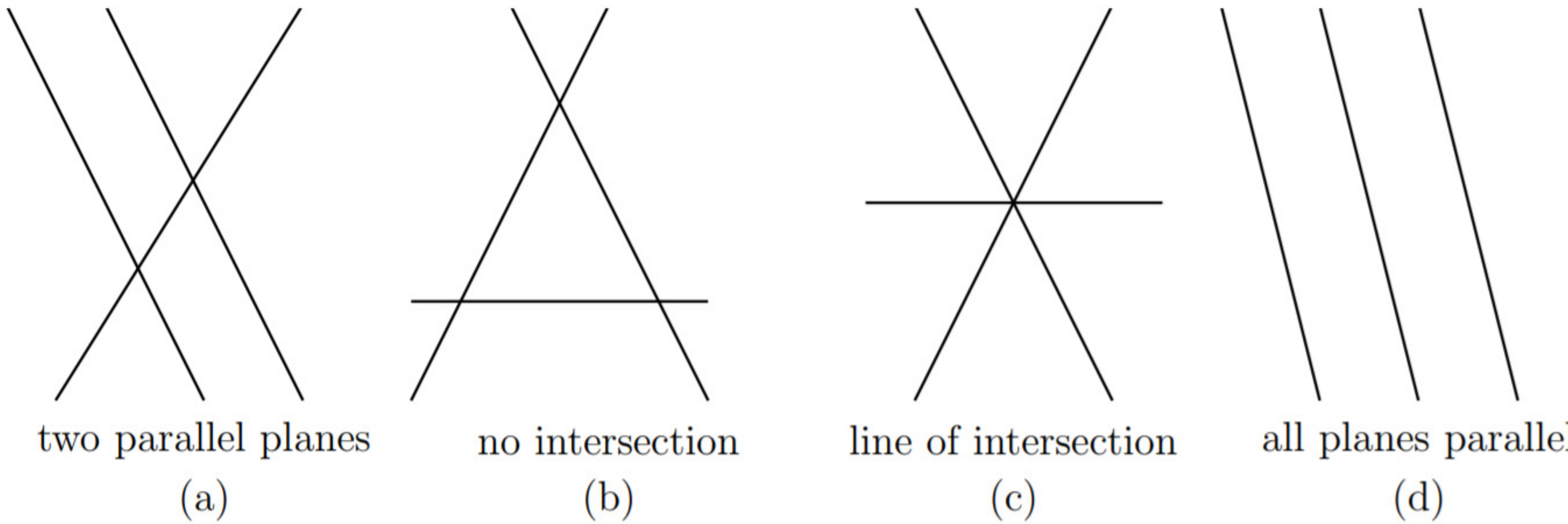
$$u = 1, v = 1, w = 2$$

With n equations in n unknowns (n-dimension),

There are n planes in the ***row picture.***

There are n vectors in the ***column picture***, plus a vector b on the right hand side.
The equations ask for a **linear combination of the n columns that equals b.**

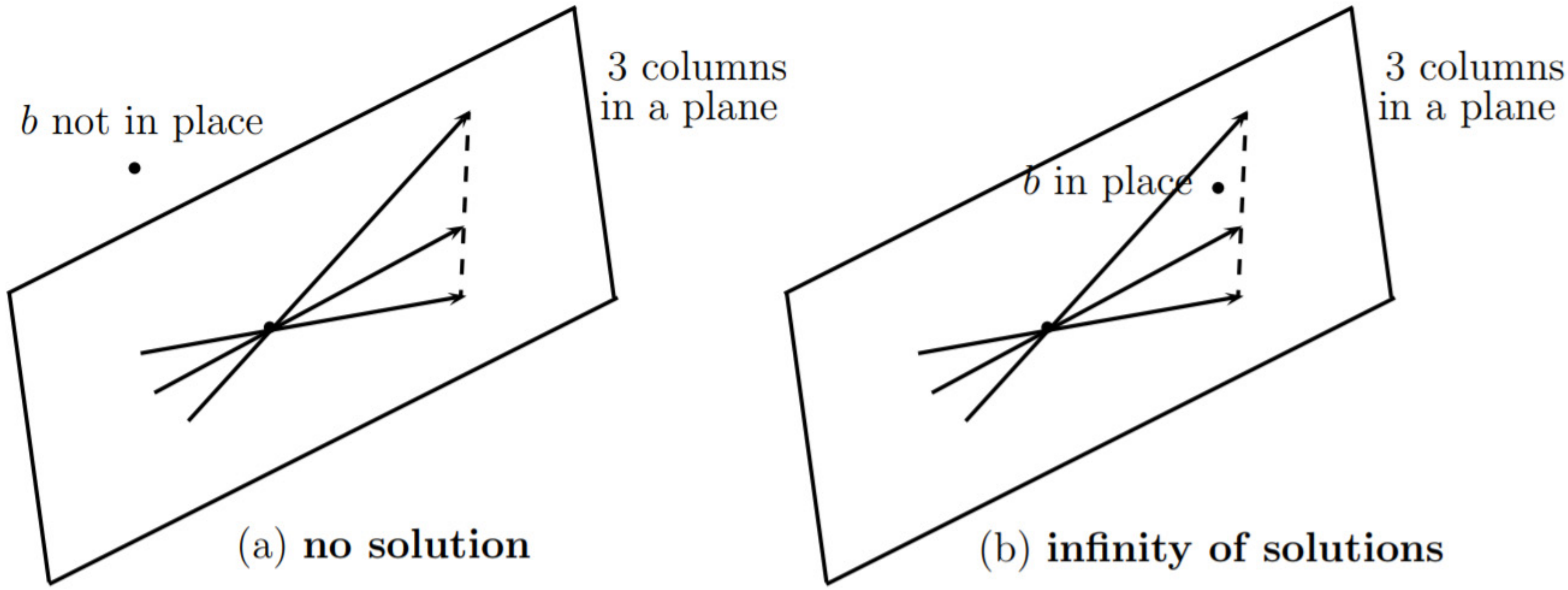
Singular cases



Singular cases: **no solution** for (a), (b), or (d),
an **infinity of solutions** for (c).

3-dimensional planes have been visualised in 2-dimension for better understanding.

Singular cases



Singular cases: b outside or inside the plane with all three columns.

If the n planes have no point in common, or infinitely many points, then the n columns lie in the same plane.