Matrices

CS6015 : Linear Algebra

Matrix : Definition

- A **matrix** is a rectangular array of numbers and each of the numbers in the matrix is called an **entry**.
- The **size (dimension)** of a matrix with n rows and m columns is denoted by $n \times m$. In denoting the size of a matrix we always list the number of rows first and the number of columns second.
- Example :

$$\begin{bmatrix} 4 & 3 & 0 & 6 & -1 & 0 \\ 0 & 2 & -4 & -7 & 1 & 3 \\ -6 & 1 & 15 & \frac{1}{2} & -1 & 0 \end{bmatrix}$$

Matrix of size (dimension) 3×4

Matrix : example

$$\begin{bmatrix} 12 \\ -4 \\ 2 \\ -17 \end{bmatrix}$$

Matrix of size (dimension) 4×1 (Column Vector)

$$\begin{bmatrix} 3 & -1 & 12 & 0 & -9 \end{bmatrix}$$

Matrix of size (dimension) 1×5 (Row Vector)

- We will often need to refer to specific entries in a matrix and so we'll need a notation to take care of that. The entry in the $\boldsymbol{i}^{\text{th}}$ row and $\boldsymbol{j}^{\text{th}}$ column of the matrix $\boldsymbol{A}$ is denoted by, $\boldsymbol{a_{ij}}$.
- The lower case letter we use to denote the entries of a matrix will always match with the upper case letter we use to denote the matrix. So the entries of the matrix $\boldsymbol{B}$ will be denoted by $\boldsymbol{b_{ij}}$.

Terminologies

- Here are some important terminologies, with examples, related to matrices

a. Main diagonal

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

b. Identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. Zero matrix

$$0_{2\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix Arithmetic and Operation

- **Equality:** $A = B$ provided *dimensions of A and B are equal* and $a_{ij} = b_{ij}$ for all i and j.
  Matrices of different sizes cannot be equal.
- **Addition, Subtraction:** $A_{n\times m} \pm B_{n\times m} = [a_{ij} \pm b_{ij}]$. *Matrices of different sizes cannot be added or subtracted.*
- **Scalar Multiple:** $cA = [ca_{ij}]$; c is any number.
- **Multiplication:** $A_{n\times p} * B_{p\times m} = A.B_{n\times m}$
- **Transpose:** $A = [a_{ij}]_{n\times m}$ then $A^T = [a_{ji}]_{m\times n} \forall i, j$
- **Trace:** $tr(A) = \sum_{i=1}^{n} a_{ii}$. *If A is not square then trace is not defined.*

Properties of Matrix Arithmetic and the Transpose

- $A + B = B + A$
- $A + (B + C) = (A + B) + C$
- $A(BC) = (AB)C$
- $A(B \pm C) = AB \pm AC$
- $(B \pm C)A = BA \pm CA$
- $a(B \pm C) = aB \pm aC$
- $(a \pm b)C \ = aC \pm bC$
- $(ab)C = a(bC)$
- $a(BC) = (aB)C = B(aC)$
- A (B) $\neq B\ (A)$, in general.

Letters in caps define matrices, while that in small denote scalars.

Properties of Matrix Arithmetic and the Transpose

- $A + 0 = 0 + A = A$
- $A - A = 0$
- $0 - A = A$
- $0A = 0 \; and \; A0 = 0$
- $A^n A^m = A^{n+m}$
- $(A^n)^m = A^{nm}$
- $(A^T)^T = A$
- $(A \pm B)^T = A^T \pm B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$

Letters in caps define matrices, while that in small denote scalars.

Inverse of square matrix

- If A is a square matrix of size n and we can find another matrix of the same size, say B, such that
$$\boldsymbol{AB = BA = I_n}$$
- Then we call ***A*** **invertible** and we say that ***B*** is an **inverse** of the matrix ***A***.
- We will denote the inverse as $\boldsymbol{A^{-1}}$.
- Not every matrix has an inverse.
  - A square matrix that has an inverse is said to be nonsingular.
  - A square matrix that does not have an inverse is said to be singular.

Important properties of the inverse matrix

Suppose that *A* and *B* are invertible matrices of the same size. Then,

a) *AB* is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

$b)$ A^{-1} is invertible and $(A^{-1})^{-1} = A$

c) For $n = 0,1,2 \ldots A^n$ is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$

d) If c is any non zero scalar then cA is invertible

and $(cA)^{-1} = \frac{1}{c}A^{-1}$.

$e)$ A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Inverse Calculation

- The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

  will be ***invertible*** if $ad - bc \neq 0$

  and ***singular*** if $ad - bc = 0$.

- If the matrix is invertible its inverse will be,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Special Matrices : Diagonal Matrix

- **Diagonal Matrix:** A square matrix is called **diagonal** if it has the following form

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & d_n \end{bmatrix}$$

- Suppose *D* is a diagonal matrix and $d_i, i = 1, .., n$ are the entries on the main diagonal.
- If one or more of the $d_i{}'s$ are zero then the matrix is singular.

Diagonal Matrix (contd.)

- On the other hand if $d_i \neq 0, \forall i$ then the matrix is invertible and the inverse is,

$$\boldsymbol{D^{-1}} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{d_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{d_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \frac{1}{d_n} \end{bmatrix}$$

Triangular matrix

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}_{n\times n}$$

Upper Triangular Matrix

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}_{n\times n}$$

Lower Triangular Matrix

- *If A is a triangular matrix with main diagonal entries $a_{11}, a_{22}, \dots, a_{nn}$ then if one or more of the a_{ii}'s are zero the matrix will be* ***singular****.*
- *On the other hand if $a_{ii} \neq 0 \ \forall i$ then the matrix is* ***invertible****.*

Symmetric and anti-symmetric matrices

Suppose that A is an $n \times m$ matrix, then A will be called **symmetric** if $A = AT$.

Some properties of symmetric matrices are:

a) For any matrix A, both AA^T and A^TA are symmetric.

b) If A is an invertible symmetric matrix then A^{-1} is also symmetric.

c) If A is invertible then AA^T and A^TA are both invertible.

Anti-Symmetric or Skew-Symmetric:

An anti-symmetric matrix is a square matrix that satisfies the identity $\mathbf{A = -A^{T}}$.

Other Special forms of matrices:

- Toeplitz matrix
- Block Circulant Matrix
- Orthogonal (also, -skew -sym)
- PD, PSD, …
- Tri-diagonal system
- Hessian
- Jacobian
- Adjoint and Adjugate matrices
- (skew-) Hermitian (or self-adjoint) matrix
- Covariance matrix
- Periodic matrices
- Compound Matrix
- g-inv & Pseudo-inv
- GRAM matrix
- Kernel of matrix
- Schur Complement
- PERM (n)
- Skew-symmetric
- DFT Matrix
- Idempotent Matrices
- Vandermonde Matrices