Matrix multiplication
LARP/2018

ACK : 1. Math slides from Henry County Schools
2. Linear Algebra and Its Applications - Gilbert Strang

Matrix Multiplication

1. The <u>order</u> makes a difference…AB is different from BA.
2. **Rule :** The number of columns in first matrix <u>must</u> equal number of rows in second matrix.

   In other words, the **inner dimensions** must be equal.
3. **Dimension of product :** **<u>The answer</u>** will be number of rows in first matrix <u>by</u> number of columns in second matrix.

   In other words, the **outer dimensions.**

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \square \end{bmatrix}$$

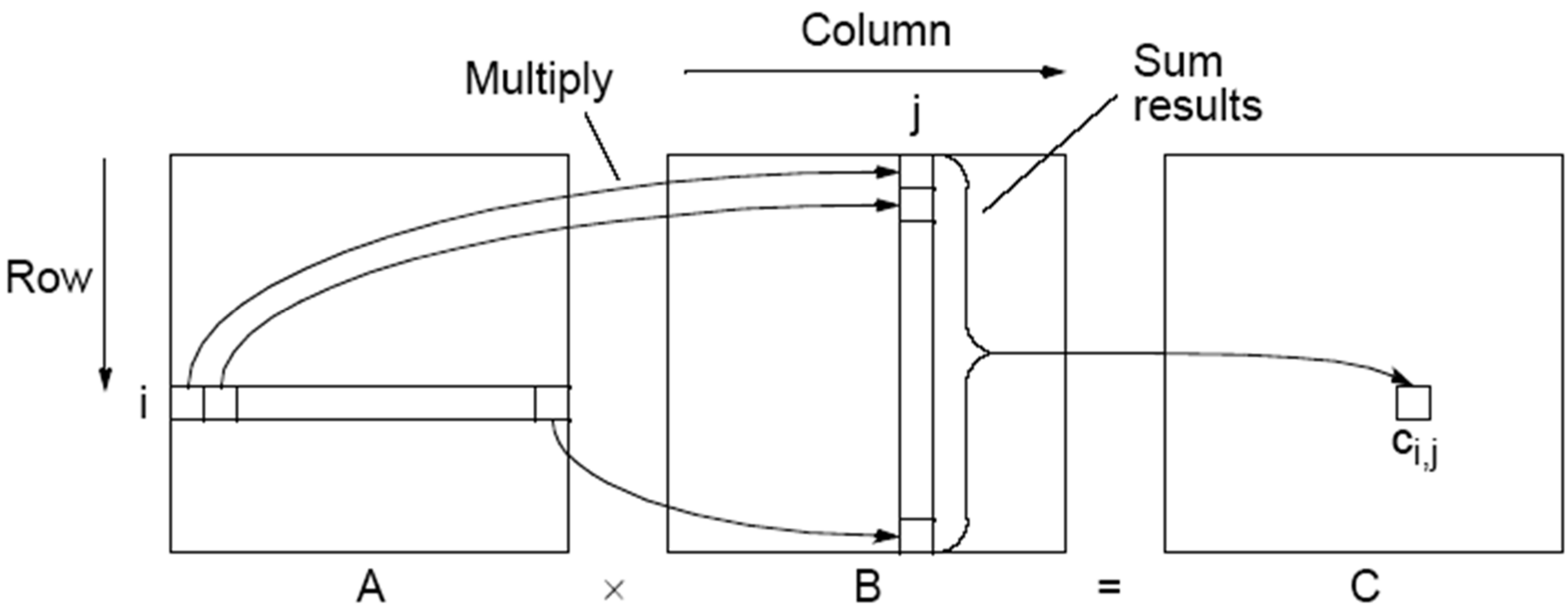
2 x 1 1 x 2 1 x 2 2 x 1

Matrix Multiplication

Multiplication of two matrices, **A** and **B**, produces the matrix **C** whose elements, $c_{i,j}$ $(0 <= i < n, 0 <= j < m)$, are computed as follows:

$$c_{i,j} = \sum_{k=0}^{l-1} a_{i,k} b_{k,j}$$

where **A** is an $n \times p$ matrix and **B** is an $p \times m$ matrix.



Are the following matrix multiplications possible?

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 5 \end{bmatrix}$$ NO

3 x 2 3 x 2

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 4 \\ 5 & 8 \end{bmatrix}$$ YES

2 x 3 3 x 2

$$\begin{bmatrix} 7 & 0 \\ 2 & \pi \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$ YES

3 x 2 2 x 3

$$\begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$$ YES

3 x 1 1 x 3

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$$ YES

2 x 2 2 x 2

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 7 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 2 & 7 \end{bmatrix}$$ NO

3 x 3 2 x 2

What is the dimension of the following products, if possible?

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 5 \end{bmatrix}$$

NO

3 x 2 3 x 2

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 4 \\ 5 & 8 \end{bmatrix}$$

YES

2 x 2

2 x 3 3 x 2

$$\begin{bmatrix} 7 & 0 \\ 2 & \pi \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$

YES

3 x 3

3 x 2 2 x 3

$$\begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$$

YES

3 x 3

3 x 1 1 x 3

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$$

YES

2 x 2

2 x 2 2 x 2

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 7 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 2 & 7 \end{bmatrix}$$

NO

3 x 3 2 x 2

The product of two matrices is found by multiplying the corresponding elements in each <u>row</u> by each <u>column</u> and then adding them together.

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

$$(1)(0) + (2)(1) + (4)(5) = 22$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 22 & \square \\ 54 & \square \end{bmatrix}$$

$$(1)(1) + (2)(4) + (4)(8) = 41$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 22 & \square \\ \square & \square \end{bmatrix}$$

$$(3)(0) + (9)(1) + (9)(5) = 54$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 22 & 41 \\ 54 & \square \end{bmatrix}$$

$$(3)(1) + (9)(4) + (9)(8) = 111$$

$$\begin{bmatrix} 22 & 41 \\ 54 & 111 \end{bmatrix}$$

Matrix Notation and Matrix Multiplication

Nine co-efficients
Three unknowns
Three right-hand sides

$$\begin{aligned} 2u + v \;+\; w &= 5 \\ 4u - 6v \qquad &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

$$Ax = b$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \qquad x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Co-efficient matrix   Solution vector   constant vector

Multiplication of a matrix and a vector

Matrix form

$Ax = b$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Row times column $\begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = [5]$

Inner product

Outer product ??

There are two ways to multiply a matrix A and a vector x.

- One way is a row at a time, each row of A combines with x to give a component of Ax. There are three inner products when A has three rows:

$$\textbf{Ax by rows} \qquad \begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 1\cdot 5 + 6\cdot 0 \\ 3\cdot 2 + 0\cdot 5 + 3\cdot 0 \\ 1\cdot 2 + 1\cdot 5 + 4\cdot 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

- Second way, multiplication a column at a time. The product Ax is found all at once, as a combination of the three columns of A:

$$\textbf{Ax by columns} \qquad 2\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

Properties of matrix multiplication

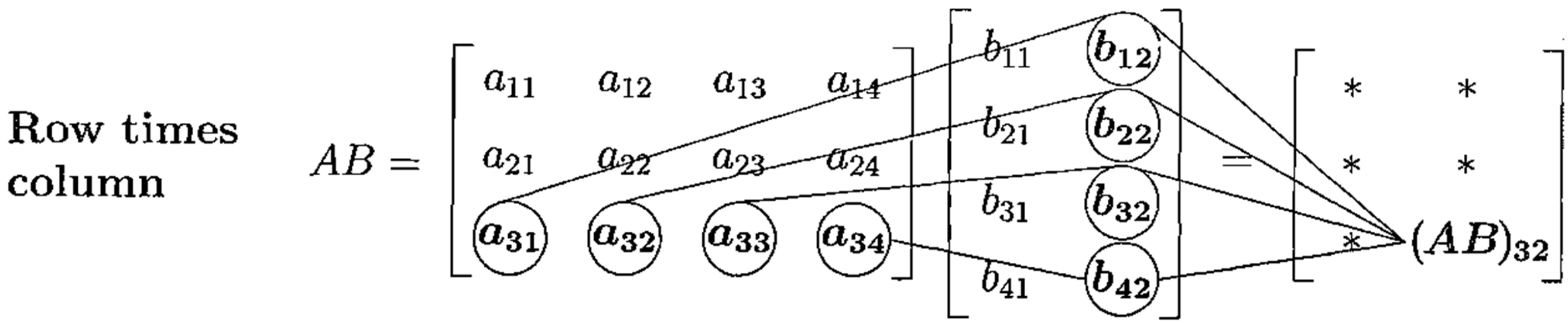
- Every product Ax can be found using whole columns. Therefore Ax is a combination of the columns of A. The coefficients are the components of x.
- The identity matrix I, with 1s on the diagonal and 0s everywhere else, leaves every vector unchanged.

Identity matrix $IA = A$ and $BI = B$.

Properties of matrix multiplication

- The i, j entry of AB is the inner product of the i-th row of A and the j-th column of B

$$(AB)_{32} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42}$$



Properties of matrix multiplication

- Each entry of AB is the product of a row and a column:
  $$(AB)_{ij} = (row\ i\ of\ A)\ times\ (column\ j\ of\ B)$$
- Each column of AB is the product of a matrix and a column:
  $$column\ j\ of\ AB = A\ times\ (column\ j\ of\ B)$$
- Each row of AB is the product of a row and a matrix:
  $$row\ i\ of\ AB = (row\ i\ of\ A)\ times\ B$$

Properties of matrix multiplication

- For matrices A, B, C, D, E and F,
- Matrix multiplication is **associative**:

$$(AB)C = A(BC)$$

- Matrix operations are **distributive**:

$$A(B+C) = AB + AC \text{ and } (B+C)D = BD + CD$$

- Matrix multiplication is **not commutative**: Usually

$$FE \neq \mathrm{EF}$$

Exception :

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -\mathbf{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{1} & 0 & 1 \end{bmatrix} \qquad EF = \begin{bmatrix} 1 & 0 & 0 \\ -\mathbf{2} & 1 & 0 \\ \mathbf{1} & 0 & 1 \end{bmatrix} = FE.$$

Commutative property of Matrix Multiplication

$$R_1 = \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) \\ -\sin(\theta_1) & \cos(\theta_1) \end{bmatrix};$$

$$\boldsymbol{R_1}.\boldsymbol{R_2} = \ ??$$

$$R_2 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix};$$

$$\boldsymbol{R_2}.\boldsymbol{R_1} = \ ??$$

$$R_1 = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix};$$

$$R_2 = ??$$

Things to ponder:

Is the computational time for matrix-sequence multiplication
Invariant to the pairings chosen to perform operations:

$$((\boldsymbol{A}(\boldsymbol{B}\ \boldsymbol{C})\boldsymbol{D})\ (\boldsymbol{E}\ (\boldsymbol{F}\ \boldsymbol{G})\boldsymbol{H})\ \ldots\ldots..$$

$$((\boldsymbol{A}\ \boldsymbol{B}\ \boldsymbol{C}\ \boldsymbol{D})(\ \boldsymbol{E}\ \boldsymbol{F})(\ \boldsymbol{G}\ \boldsymbol{H}\ldots\ldots\ldots\ldots.$$

$$(\boldsymbol{A}(\ \boldsymbol{B}(\ \boldsymbol{C}(\ \boldsymbol{D}\ (\boldsymbol{E}\ (\boldsymbol{F}\ (\boldsymbol{G}\ (\boldsymbol{H}\ldots\ldots.$$

Is there any optimal pairing of the sequence ??