Inverses and Tranposes

LARP / 2018

ACK : Linear Algebra and Its Applications - Gilbert Strang

Inverse matrix

- The inverse of an n by n matrix is another n by n matrix. The inverse of A is written A^{-1} (and pronounced "A inverse").

- The fundamental property is simple: If you multiply by A and then multiply by A^{-1} , you are back where you started:

Inverse matrix If $b = Ax$ then $A^{-1}b = x$

- Thus $A^{-1}Ax = x$. The matrix A^{-1} times A is the identity matrix. ***Not all matrices have inverses. An inverse is impossible when*** Ax ***is zero and*** x ***is nonzero.*** Then A^{-1} would have to get back from $Ax = 0$ to x. No matrix can multiply that zero vector Ax and produce a nonzero vector x.

- Our goals are to define the inverse matrix and compute it and use it, when A^{-1} exists—and then to understand which matrices don't have inverses.

Properties : Inverse matrix

1K The **inverse** of A is a matrix B such that $BA = I$ and $AB = I$. There is at most one such B, and it is denoted by A^{-1}:

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I. \tag{1}$$

Note* 1.** ***The inverse exists if and only if elimination produces n pivots (row exchanges allowed). Elimination solves $Ax = b$ without explicitly finding A^{-1}.

***Note* 2.** The matrix A cannot have two different inverses, Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this "proof by parentheses":

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{which is} \quad B = C. \tag{2}$$

This shows that a *left-inverse* B (multiplying from the left) and a *right-inverse* C (multiplying A from the right to give $AC = I$) must be the *same matrix*.

***Note* 3.** If A is invertible, the one and only solution to $Ax = b$ is $x = A^{-1}b$:

$$\textbf{\textit{Multiply}} \quad Ax = b \quad \textbf{\textit{by}} \quad A^{-1}. \quad \textbf{\textit{Then}} \quad x = A^{-1}Ax = A^{-1}b.$$

Note* 4.** (Important) ***Suppose there is a nonzero vector x ***such that*** $Ax = 0$. *Then* A *cannot have an inverse*. To repeat: No matrix can bring 0 back to x.

If A is invertible, then $Ax = 0$ can only have the zero solution $x = 0$.

Properties : Inverse matrix

***Note* 5.** A 2 by 2 matrix is invertible if and only if $ad-bc$ is not zero:

$$\textbf{2 by 2 inverse} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3}$$

This number $ad-bc$ is the *determinant* of A. A matrix is invertible if its determinant is not zero (Chapter 4). In MATLAB, the invertibility test is *to find n nonzero pivots*. Elimination produces those pivots before the determinant appears.

***Note* 6.** A diagonal matrix has an inverse provided no diagonal entries are zero:

$$\text{If} \quad A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad \text{then} \quad A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix} \quad \text{and} \quad AA^{-1} = I.$$

When two matrices are involved, not much can be done about the inverse of $A+B$. The sum might or might not be invertible. Instead, it is the inverse of their *product* AB which is the key formula in matrix computations. Ordinary numbers are the same: $(a+b)^{-1}$ is hard to simplify, while $1/ab$ splits into $1/a$ times $1/b$. But for matrices *the order of multiplication must be correct*—if $ABx=y$ then $Bx=A^{-1}y$ and $x=B^{-1}A^{-1}y$. **The inverses come in reverse order**.

Properties : Inverse matrix

1L A product AB of invertible matrices is inverted by $B^{-1}A^{-1}$:

$$\textbf{Inverse of } AB \qquad (AB)^{-1} = B^{-1}A^{-1}. \tag{4}$$

Proof. To show that $B^{-1}A^{-1}$ is the inverse of AB, we multiply them and use the associative law to remove parentheses. Notice how B sits next to B^{-1}:

$$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I.$$

□

A similar rule holds with three or more matrices:

$$\textbf{Inverse of } ABC \qquad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

We saw this change of order when the elimination matrices E, F, G were inverted to come back from U to A. In the forward direction, $GFEA$ was U. In the backward direction, $L = E^{-1}F^{-1}G^{-1}$ was the product of the inverses. *Since G came last, G^{-1} comes first*. Please check that A^{-1} would be $U^{-1}GFE$.

Calculation of A^{-1} : The Gauss-Jordan Method

- Given the $n \times n$ matrix A:

    1. Adjoin the $n \times n$ identity matrix I to obtain the augmented matrix $[A \mid I]$.

    2. Use a sequence of row operations to reduce $[A \mid I]$ to the form $[I \mid B]$ if possible.

- Then the matrix B is the inverse of A.

Example

- Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

Solution

- We form the augmented matrix

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Example

- Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

<u>Solution</u>

- And use the Gauss-Jordan elimination method to reduce it to the form $[I \mid B]$:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\boldsymbol{R_1 - R_2}} \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\begin{array}{l} \boldsymbol{-R_1} \\ \boldsymbol{R_2 + 3R_3} \\ \boldsymbol{R_3 + 2R_1} \end{array}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & -1 & 2 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{\begin{array}{c} \boldsymbol{R_1 + R_2} \\ \boldsymbol{-R_2} \\ \boldsymbol{R_3 - R_2} \end{array}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Example

- Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

<u>Solution</u>

- And use the Gauss-Jordan elimination method to reduce it to the form $[I \mid B]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[\boldsymbol{R_2 + R_3}]{\boldsymbol{R_1 - R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Previous step

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\boldsymbol{I_n}} \quad \underbrace{\begin{matrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{matrix}}_{\boldsymbol{B}}$$

$$B = A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Remarks : Gauss-Jordan

***Remark* 1.** In spite of this brilliant success in computing A^{-1}, I don't recommend it, I admit that A^{-1} solves $Ax = b$ in one step. Two triangular steps are better:

$$x = A^{-1}b \quad \text{separates into} \quad Lc = b \quad \text{and} \quad Ux = c.$$

We could write $c = L^{-1}b$ and then $x = U^{-1}c = U^{-1}L^{-1}b$. But note that we did not explicitly form, and in actual computation *should not form*, these matrices L^{-1} and U^{-1}.

It would be a waste of time, since we only need back-substitution for x (and forward substitution produced c).

A similar remark applies to A^{-1}; the multiplication $A^{-1}b$ would still take n^2 steps. ***It is the solution that we want, and not all the entries in the inverse***.

Remarks : Gauss-Jordan

Remark 2. Purely out of curiosity, we might count the number of operations required to find A^{-1}. The normal count for each new right-hand side is n^2, half in the forward direction and half in back-substitution. With n right-hand sides $e_1, \ldots, e_n$ this makes n^3. After including the $n^3/3$ operations on A itself, the total seems to be $4n^3/3$.

This result is a little too high because of the zeros in the e_j. Forward elimination changes only the zeros below the 1. This part has only $n - j$ components, so the count for e_j is effectively changed to $(n-j)^2/2$. Summing over all j, the total for forward elimination is $n^3/6$. This is to be combined with the usual $n^3/3$ operations that are applied to A, and the $n(n^2/2)$ back-substitution steps that finally produce the columns x_j of A^{-1}. *The final count of multiplications for computing A^{-1} is n^3:*

$$\textbf{Operation count} \qquad \frac{n^3}{6} + \frac{n^3}{3} + n\left(\frac{n^2}{2}\right) = n^3.$$

This count is remarkably low. Since matrix multiplication already takes n^3 steps, it requires as many operations to compute A^2 as it does to compute A^{-1}! That fact seems almost unbelievable (and computing A^3 requires twice as many, as far as we can see). Nevertheless, if A^{-1} is not needed, it should not be computed.

Remarks : Gauss-Jordan

Remark 3. In the Gauss-Jordan calculation we went all the way forward to U, before starting backward to produce zeros above the pivots. That is like Gaussian elimination, but other orders are possible. We could have used the second pivot when we were there earlier, to create a zero above it as well as below it. This is not smart. At that time the second row is virtually full, whereas near the end it has zeros from the upward row operations that have already taken place.

Finding the inverse of a square matrix using LU decomposition

The inverse [B] of a square matrix [*A*] is defined as

How can LU Decomposition be used to find the inverse?

Assume the first column of [*B*] to be $[b_{11}\ b_{12}\ \ldots\ b_{n1}]^T$

Using this and the definition of matrix multiplication

Finding the inverse of a square matrix

First column of [*B*]

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of [*B*]

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in [*B*] can be found in the same manner.

Example: Inverse of a Matrix

Find the inverse of a square matrix [A]

$$[A]=\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Using the decomposition procedure, the [L] and [U] matrices are found to be

$$[A]=[L][U]=\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example: Inverse of a Matrix

Solving for the each column of [*B*] requires two steps

1)Solve [*L*] [*Z*] = [*C*] for [*Z*]

2)Solve [*U*] [*X*] = [*Z*] for [*X*]

Step 1: $$[L][Z]=[C] \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This generates the equations:

$$\begin{aligned} z_1 &= 1 \\ 2.56 z_1 + z_2 &= 0 \\ 5.76 z_1 + 3.5 z_2 + z_3 &= 0 \end{aligned}$$

Example: Inverse of a Matrix

Solving for $[Z]$

$$
\begin{aligned}
z_1 &= 1 \\
z_2 &= 0 - 2.56 z_1 \\
&= 0 - 2.56(1) \\
&= -2.56 \\
z_3 &= 0 - 5.76 z_1 - 3.5 z_2 \\
&= 0 - 5.76(1) - 3.5(-2.56) \\
&= 3.2
\end{aligned}
$$

$$
[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}
$$

Example: Inverse of a Matrix

Solving $[U][X] = [Z]$ for $[X]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$\begin{aligned} 25b_{11} + 5b_{21} + b_{31} &= 1 \\ -4.8b_{21} - 1.56b_{31} &= -2.56 \\ 0.7b_{31} &= 3.2 \end{aligned}$$

Example: Inverse of a Matrix

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560\, b_{31}}{-4.8}$$

$$= \frac{-2.56 + 1.560\left(4.571\right)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25}$$

$$= \frac{1 - 5\left(-0.9524\right) - 4.571}{25} = 0.04762$$

So the first column of the inverse of [*A*] is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Example: Inverse of a Matrix

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Example: Inverse of a Matrix

The inverse of $[A]$ is

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

Invertible = Nonsingular (n pivots)

Suppose A has a full set of n pivots. $AA^{-1} = I$ gives n separate systems $Axi = ei$ for the columns of A^{-1} . They can be solved by elimination or by Gauss-Jordan. Row exchanges may be needed, but the columns of A^{-1} are determined.

Strictly speaking, we have to show that the matrix A^{-1} with those columns is also a left-inverse. Solving $AA^{-1} = I$ has at the same time solved $A^{-1}A = I$, but why? **A 1-sided inverse of a square matrix is automatically a 2-sided inverse.** To see why, notice that *every Gauss-Jordan step is a multiplication on the left by an elementary matrix*. We are allowing three types of elementary matrices:

1. E_{ij} to subtract a multiple ℓ of row j from row i
2. P_{ij} to exchange rows i and j
3. D (or D^{-1}) to divide all rows by their pivots.

The Gauss-Jordan process is really a giant sequence of matrix multiplications:

$$(D^{-1} \cdots E \cdots P \cdots E)A = I. \tag{6}$$

That matrix in parentheses, to the left of A, is evidently a left-inverse! It exists, it equals the right-inverse by $Note\ 2$ (slide no. 3), so every nonsingular matrix is invertible.

Invertible = Nonsingular (n pivots) (contd.)

The converse is also true: **If A is invertible, it has n pivots.** In an extreme case that is clear: A cannot have a whole column of zeros. The inverse could never multiply a column of zeros to produce a column of I. In a less extreme case, suppose elimination starts on an invertible matrix A but breaks down at column 3:

Breakdown
No pivot in column 3

$$A' = \begin{bmatrix} d_1 & x & x & x \\ 0 & d_2 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \end{bmatrix}.$$

This matrix cannot have an inverse, no matter what the x's are. One proof is to use column operations (for the first time?) to make the whole third column zero. By subtracting multiples of column 2 and then of column 1, we reach a matrix that is certainly not invertible. Therefore the original A was not invertible. Elimination gives a complete test: ***An n by n matrix is invertible if and only if it has n pivots.***

Transpose Matrix

- The transpose of A is denoted by A^T . Its columns are taken directly from the rows of A—the i^{th} row of A becomes the i^{th} column of A^T :

$$\textbf{Transpose} \qquad \text{If} \quad A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{then} \quad A^{\mathrm{T}} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}.$$

At the same time the columns of A become the rows of A^T , If A is an m by n matrix, then A^T is n by m. The final effect is to flip the matrix across its main diagonal, and the entry in row i, column j of A^T comes from row j, column i of A:

$$\textbf{Entries of } A^{\mathrm{T}} \qquad (A^{\mathrm{T}})_{ij} = A_{ji}. \tag{7}$$

- The transpose of a lower triangular matrix is upper triangular. The transpose of A^T brings us back to A.
- If we add two matrices and then transpose, the result is the same as first transposing and then adding: $(A + B)^T$ is the same as $A^T + B^T$.

Properties : Transpose Matrix

1M

(i) The transpose of AB is $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$,

(ii) The transpose of A^{-1} is $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$.

$$\textbf{Start from} \qquad AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 5 & 5 & 5 \end{bmatrix}$$

$$\textbf{Transpose to} \qquad B^{\mathrm{T}}A^{\mathrm{T}} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 3 & 5 \\ 3 & 5 \end{bmatrix}.$$

To establish the formula for $(A^{-1})^{\mathrm{T}}$, start from $AA^{-1} = I$ and $A^{-1}A = I$ and take transposes. On one side, $I^{\mathrm{T}} = I$. On the other side, we know from part (i) the transpose of a product. You see how $(A^{-1})^{\mathrm{T}}$ is the inverse of A^{T}, proving (ii):

$$\textbf{Inverse of } A^{\mathrm{T}} = \textbf{Transpose of } A^{-1} \qquad (A^{-1})^{\mathrm{T}}A^{\mathrm{T}} = I. \tag{8}$$

Symmetric matrix

- **A symmetric matrix is a matrix that equals its own transpose**: $A^T = A$. The matrix is necessarily square. Each entry on one side of the diagonal equals its "**mirror image**" on the other side: $a_{ij} = a_{ji}$. Two simple examples are A and D (and also A^{-1}):

$$\textbf{Symmetric matrices} \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 8 & -2 \\ -2 & 1 \end{bmatrix}.$$

- A symmetric matrix need not be invertible; it could even be a matrix of zeros.
- But if A^{-1} exists it is also symmetric.
- From formula (ii) (previous slide), the transpose of A^{-1} always equals $(A^T)^{-1}$; for a symmetric matrix this is just A^{-1}. A^{-1} equals its own transpose; it is symmetric whenever A is.

Symmetric Products $R^T R$, RR^T , and LDL^T

Choose any matrix R, probably rectangular. Multiply R T times R. Then the product R TR is automatically a square symmetric matrix:

The transpose of $\boldsymbol{R^T R}$ is $\boldsymbol{R^T (R^T)^T}$, which is $\boldsymbol{R^T R}$.

That is a quick proof of symmetry for $R^T R$. Its i, j entry is the inner product of row i of R^T (column i of R) with column j of R. The (j, i) entry is the same inner product, column j with column i. So $R^T R$ is symmetric. RR^T is also symmetric, but it is different from $R^T R$. In my experience, most scientific problems that start with a rectangular matrix R end up with $R^T R$ or RR^T or both.

Example:

$$R = [1 \ \ 2] \text{ and } R^{\mathrm{T}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ produce } R^{\mathrm{T}} R = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ and } RR^{\mathrm{T}} = [5].$$

The product $R^T R$ is $n \ by \ n$. In the opposite order, RR^T is $m \ by \ m$. Even if $m = n$, it is not very likely that $R^T R = RR^T$. Equality can happen, but it's not normal.

1N *Suppose $A = A^T$ can be factored into $A = LDU$ without row exchanges. Then U is the transpose of L. The symmetric factorization becomes $A = LDL^T$.*

The transpose of $A = LDU$ gives $A^T = U^T D^T L^T$. Since $A = A^T$, we now have two factorizations of A into lower triangular times diagonal times upper triangular. (L^T is upper triangular with ones on the diagonal, exactly like U.) Since the factorization is unique, L^T must be identical to U.

$$L^{\mathrm{T}} = U \textbf{ and } A = LDL^{\mathrm{T}} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = LDL^{\mathrm{T}}.$$

When elimination is applied to a symmetric matrix, $A^T = A$ is an advantage. The smaller matrices stay symmetric as elimination proceeds, and we can work with half the matrix! The lower right-hand corner remains symmetric:

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c \\ 0 & d - \frac{b^2}{a} & e - \frac{\mathbf{bc}}{\mathbf{a}} \\ 0 & e - \frac{\mathbf{bc}}{\mathbf{a}} & f - \frac{c^2}{a} \end{bmatrix}.$$

The work of elimination is reduced from $n^3/3$ to $n^3/6$. There is no need to store entries from both sides of the diagonal, or to store both L and U.