Triangular Factors and Row Exchanges

LARP / 2018

ACK : 1. Linear Algebra and Its Applications - Gilbert Strang
2. Autar Kaw, Transforming Numerical Methods Education for STEM Graduates

We want to look again at elimination, to see what it means in terms of matrices. The starting point was the model system $Ax = b$:

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b. \tag{1}$$

Then there were three elimination steps, with multipliers 2, −1, −1:

Step 1. Subtract 2 times the first equation from the second;
Step 2. Subtract −1 times the first equation from the third;
Step 3. Subtract −1 times the second equation from the third.

Forward elimination

The result was an equivalent system $Ux = c$, with a new coefficient matrix U:

Upper triangular

$$Ux = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} = c. \tag{2}$$

This matrix U is **upper triangular**—all entries below the diagonal are zero.
The new right side c was derived from the original vector b by the same steps that took A into U.
$Ux = c$ is solved by back-substitution. Here we concentrate on connecting A to U.

Elementary matrices corresponding to step 1, 2 and 3 are :

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -\mathbf{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{1} & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathbf{1} & 1 \end{bmatrix}$$

Results of all three steps : $GFEA = U$

From A to U $\quad GFE = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & 1 & 1 \end{bmatrix}$

$$\underset{G}{\begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix}} \underset{F}{\begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}$$

How would we get from U back to A? **How can we undo the steps of Gaussian elimination?**

Instead of subtracting, we add twice the first row to the second. (Not twice the second row to the first!) The result of doing both the subtraction and the addition is to bring back the identity matrix:

Inverse of subtraction is addition

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

One operation cancels the other. In matrix terms, one matrix is the ***inverse*** of the other. If the elementary matrix E has the number $-\ell$ in the (i, j) position, then its inverse E^{-1} has $+\ell$ in that position. Thus $E^{-1}E = I$, which is equation (4).

We can invert each step of elimination, by using E^{-1} and F^{-1} and G^{-1}. I think it's not bad to see these inverses now, before the next section. The final problem is to undo the whole process at once, and see what matrix takes U back to A.

Since step 3 was last in going from A to U, its matrix G must be the first to be inverted in the reverse direction. Inverses come in the opposite order! The second reverse step is F^{-1} and the last is E^{-1} :

$$\textbf{From } U \textbf{ back to } A \qquad E^{-1}F^{-1}G^{-1}U = A \text{ is } LU = A.$$

You can substitute $GFEA$ for U, to see how the inverses knock out the original steps. Now we recognize the matrix L that takes U back to A. It is called L, because it is **lower triangular**. And it has a special property that can be seen only by multiplying the three inverse matrices in the right order:

$$E^{-1}F^{-1}G^{-1} = \begin{bmatrix} 1 & & \\ \mathbf{2} & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ \mathbf{-1} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & \mathbf{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \mathbf{2} & 1 & \\ \mathbf{-1} & \mathbf{-1} & 1 \end{bmatrix} = L.$$

The special thing is that ***the entries below the diagonal are the multipliers*** $\ell = 2, -1$, and -1. When matrices are multiplied, there is usually no direct way to read off the answer. Here the matrices come in just the right order so that their product can be written down immediately. If the computer stores each multiplier ℓ_{ij}—the number that multiplies the pivot row j when it is subtracted from row i, and produces a zero in the i, j position—then these multipliers give a complete record of elimination.

The numbers ℓ_{ij} fit right into the matrix L that takes U back to A.

1H Triangular factorization $A = LU$ with no exchanges of rows. L is lower triangular, with 1s on the diagonal. The multipliers ℓ_{ij} (taken from elimination) are below the diagonal. U is the upper triangular matrix which appears after forward elimination, The diagonal entries of U are the pivots.

Example :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \text{ goes to } U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \text{ with } L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}. \quad \text{Then } LU = A.$$

Example : (which needs a row exchange)

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{cannot be factored into } A = LU.$$

Example : (with all pivots and multipliers equal to 1)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU.$$

From A to U there are subtractions of rows. From U to A there are additions of rows.

Example: (when U is the *identity* and L is the same as A)

Lower triangular case $\quad A = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}.$

The elimination steps on this A are easy:
(i) E subtracts ℓ_{21} times row 1 from row 2,
(ii) F subtracts ℓ_{31} times row 1 from row 3, and
(iii) G subtracts ℓ_{32} times row 2 from row 3.
The result is the identity matrix $U = I$. The inverses of $E, F, and\ G$ will bring back A:

$$\boxed{E^{-1} \text{ applied to } F^{-1} \text{ applied to } G^{-1} \text{ applied to } I \text{ produces } A.}$$

$$\begin{bmatrix} 1 & & \\ \ell_{21} & 1 & \\ & & 1 \end{bmatrix} \text{ times } \begin{bmatrix} 1 & & \\ & 1 & \\ \ell_{31} & & 1 \end{bmatrix} \text{ times } \begin{bmatrix} 1 & & \\ & 1 & \\ & \ell_{32} & 1 \end{bmatrix} \text{ equals } \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}.$$

The order is right for the ℓ's to fall into position. This always happens! Note that parentheses in $E^{-1}F^{-1}G^{-1}$ were not necessary because of the associative law.

$A = LU$: The n by n case

The matrix L, applied to U, brings back A:

$$A = LU \qquad \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} \text{row 1 of } U \\ \text{row 2 of } U \\ \text{row 3 of } U \end{bmatrix} = \text{original } A. \tag{7}$$

The proof is to apply the *steps of elimination*. On the right-hand side they take A to U. On the left-hand side they reduce L to I, as in previous example. (The first step subtracts ℓ_{21} times $(1,0,0)$ from the second row, which removes ℓ_{21}.) Both sides of (7) end up equal to the same matrix U, and the steps to get there are all reversible. Therefore (7) is correct and $A = LU$.

Example: (A = LU, with zeros in the empty spaces)

$$A = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}.$$

That shows how a matrix A with three diagonals has factors L and U with two diagonals.

One Linear System = Two Triangular Systems

There is a serious practical point about $A = LU$. It is more than just a record of elimination steps; L and U are the right matrices to solve $Ax = b$. In fact A could be thrown away! We go from b to c by forward elimination (this uses L) and we go from c to x by back-substitution (that uses U). We can and should do it without A:

$$\textbf{Splitting of } Ax = b \qquad \textit{First} \quad Lc = b \quad \textit{and then} \quad Ux = c. \tag{8}$$

Multiply the second equation by L to give $LUx = Lc$, which is $Ax = b$. Each triangular system is quickly solved. That is exactly what a good elimination code will do:

1. ***Factor*** (from A find its factors L and U).
2. ***Solve*** (from L and U and b find the solution x).

The separation into ***Factor*** and ***Solve*** means that a series of b's can be processed. The ***Solve*** subroutine obeys equation (8).

Example :This is the previous matrix A (in slide 8) with a right-hand side $b = (1,1,1,1)$.

$$Ax = b \qquad \begin{array}{rcrcrcrcl} x_1 & - & x_2 & & & & & = & 1 \\ -x_1 & + & 2x_2 & - & x_3 & & & = & 1 \\ & - & x_2 & + & 2x_3 & - & x_4 & = & 1 \\ & & & - & x_3 & + & 2x_4 & = & 1 \end{array} \qquad \text{splits into } Lc = b \text{ and } Ux = c.$$

$$Lc = b \qquad \begin{array}{rcrcrcrcl} c_1 & & & & & & & = & 1 \\ -c_1 & + & c_2 & & & & & = & 1 \\ & - & c_2 & + & c_3 & & & = & 1 \\ & & & - & c_3 & + & c_4 & = & 1 \end{array} \qquad \text{gives} \quad \boldsymbol{c} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{bmatrix}.$$

$$Ux = c \qquad \begin{array}{rcrcrcrcl} x_1 & - & x_2 & & & & & = & 1 \\ & & x_2 & - & x_3 & & & = & 2 \\ & & & & x_3 & - & x_4 & = & 3 \\ & & & & & & x_4 & = & 4 \end{array} \qquad \text{gives} \quad \boldsymbol{x} = \begin{bmatrix} \mathbf{10} \\ \mathbf{9} \\ \mathbf{7} \\ \mathbf{4} \end{bmatrix}.$$

Remarks

The LU form is "*unsymmetric*" on the diagonal: L has $1s$ where U has the pivots. This is easy to correct. **Divide out of U a diagonal pivot matrix D:**

$$\textbf{Factor out } D \qquad U = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \vdots \\ & 1 & u_{23}/d_2 & \vdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}. \qquad (9)$$

In the last example all pivots were $d_i = 1$. In that case $D = I$. But that was very exceptional, and normally LU is different from LDU (also written LDV).

> ***The triangular factorization can be written $A = LDU$, where L and U have $1s$ on the diagonal and D is the diagonal matrix of pivots.***

Whenever you see LDU or LDV, it is understood that U or V has is on the diagonal— each row was divided by the pivot in D. Then L and U are treated evenly. An example of LU splitting into LDU is

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ & -2 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} = LDU.$$

That has the $1s$ on the diagonals of L and U, and the pivots 1 and −2 in D.

Remarks

1l If $A = L_1D_1U_1$ and also $A = L_2D_2U_2$, where the L's are lower triangular with unit diagonal, the U's are upper triangular with unit diagonal, and the D's are diagonal matrices with no zeros on the diagonal, then $L_1 = L_2$, $D_1 = D_2$, $U_1 = U_2$. The LDU factorization and the LU factorization are uniquely determined by A.

Row Exchanges and Permutation Matrices

We now have to face a problem that has so far been avoided: The number we expect to use as a pivot might be zero. This could occur in the middle of a calculation. It will happen at the very beginning if $a_{11} = 0$. A simple example is

$$\textbf{Zero in the pivot position} \qquad \begin{bmatrix} \mathbf{0} & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

The difficulty is clear; no multiple of the first equation will remove the coefficient 3.

The remedy is equally clear. Exchange the two equations, moving the entry 3 up into the pivot. In this example the matrix would become upper triangular:

$$\textbf{Exchange rows} \qquad \begin{aligned} 3u + 4v &= b_2 \\ 2v &= b_1 \end{aligned}$$

To express this in matrix terms, we need the ***permutation matrix*** P that produces the row exchange. It comes from exchanging the rows of I:

$$\textbf{Permutation} \qquad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}.$$

P has the same effect on b, exchanging b_1 and b_2. The new system is $PAx = Pb$. The unknowns u and v are not reversed in a row exchange.

Permutation matrix

A permutation matrix P has the same rows as the identity (in some order). There is a single "1" in every row and column. The most common permutation matrix is $P = I$ (it exchanges nothing). The product of two permutation matrices is another permutation— the rows of I get reordered twice.

After $P = I$, the simplest permutations exchange two rows. Other permutations exchange more rows. There are $n! = (n)(n-1)\cdots(1)$ permutations of size n. Row 1 has n choices, then row 2 has $n-1$ choices, and finally the last row has only one choice. All 3 by 3 permutations ($3! = (3)(2)(1) = 6$ matrices) are displayed:

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \qquad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \qquad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \qquad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

There will be 24 permutation matrices of order $n = 4$.

- P^{-1} is always the same as P^T.

Permutation matrix (contd.)

A zero in the pivot location raises two possibilities: **The trouble may be easy to fix, or it may be serious**. This is decided by looking *below the zero*. If there is a non-zero entry lower down in the same column, then a row exchange is carried out. The nonzero entry becomes the needed pivot, and elimination can get going again:

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix} \qquad \begin{array}{lll} d = 0 & \Longrightarrow & \text{no first pivot} \\ a = 0 & \Longrightarrow & \text{no second pivot} \\ c = 0 & \Longrightarrow & \text{no third pivot.} \end{array}$$

If $d = 0$, the problem is incurable and this matrix is **singular**. There is no hope for a unique solution to $Ax = b$. If d is not zero, an exchange P_{13} of rows 1 and 3 will move d into the pivot. However the next pivot position also contains a zero. The number a is now below it (the e above it is useless). If a is not zero then another row exchange P_{23} is called for:

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P_{23}P_{13}A = \begin{bmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix}$$

Permutation matrix (contd.)

One more point: The permutation P23P13 will do both row exchanges at once:

$$P_{13} \textbf{ acts first} \qquad P_{23}P_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P.$$

If we had known, we could have multiplied A by P in the first place. With the rows in the right order PA, any non-singular matrix is ready for elimination.

Elimination in a Nutshell: $PA = LU$

The main point is this: If elimination can be completed with the help of row exchanges, then we can imagine that those exchanges are done first (by P). *The matrix PA will not need row exchanges*. In other words, PA allows the standard factorization into L times U. The theory of Gaussian elimination can be summarized in a few lines:

1J In the ***nonsingular*** case, there is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. Then $Ax = b$ has a *unique solution*:

With the rows reordered in advance, PA can be factored into LU.

In the ***singular*** case, no P can produce a full set of pivots: elimination fails.

In practice, we also consider a row exchange when the original pivot is near zero— even if it is not exactly zero. Choosing a larger pivot reduces the roundoff error. You have to be careful with L. Suppose elimination subtracts row 1 from row 2, creating $\ell_{21} = 1$. Then suppose it exchanges rows 2 and 3. If that exchange is done in advance, the multiplier will change to $\ell_{31} = 1$ in $PA = LU$.

Example:
$$A=\begin{bmatrix}1&1&1\\1&1&3\\2&5&8\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1\\0&\mathbf{0}&2\\0&3&6\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1\\0&3&6\\0&0&2\end{bmatrix}=U.$$

That row exchange recovers LU—but now $\ell_{31}=1$ and $\ell_{21}=2$:

$$P=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\quad\text{and}\quad L=\begin{bmatrix}1&0&0\\2&1&0\\1&0&1\end{bmatrix}\quad\text{and}\quad PA=LU. \tag{11}$$

In MATLAB, A([r k] :) exchanges row k with row r below it (where the kth pivot has been found). We update the matrices L and P the same way. At the start, $P=I$ and sign $=+1$:

```
A([r k], :) = A([k r], :);
L([r k], 1:k-1) = L([k r], 1:k-1);
P([r k], :) = P([k r], :);
sign = -sign
```

The "**sign**" of P tells whether the number of row exchanges is even $(sign=+1)$ or odd $(sign=-1)$. A row exchange reverses sign. The final value of sign is the **determinant** of P and it does not depend on the order of the row exchanges.

Summarization

A good elimination code saves L and U and P. Those matrices carry the information that originally came in A—and they carry it in a more usable form. $Ax = b$ reduces to two triangular systems. This is the practical equivalent of the calculation we do next—to find the inverse matrix A^{-1} and the solution $x = A^{-1}b$.

LU Factorization

extra slides

Introduction

- Apart from Gaussian Elimination, another way of solving a system of equations is by using a factorization technique for matrices called **LU decomposition**.
- This factorization of matrix $[A]$ involves two matrices
  - one lower triangular matrix (L)
  - one upper triangular matrix (U)

such that

- factorization methods separate the time-consuming elimination of the matrix from the manipulations of the right-hand-side
- Once has been factored (or decomposed), multiple right-hand-side vectors can be evaluated in an efficient manner.

How does LU Decomposition work?

If solving a set of linear equations	$[A][X] = [C]$
If $[A] = [L][U]$ then	$[L][U][X] = [C]$
Multiply by	$[L]^{-1}$
Which gives	$[L]^{-1}[L][U][X] = [L]^{-1}[C]$
Remember $[L]^{-1}[L] = [I]$ which leads to	$[I][U][X] = [L]^{-1}[C]$
Now, if $[I][U] = [U]$ then	$[U][X] = [L]^{-1}[C]$
Now, let	$[L]^{-1}[C]=[Z]$
Which ends with	$[L][Z] = [C]$ ... (1)
and	$[U][X] = [Z]$ ... (2)

LU Decomposition

How can this be used?

Given $[A][X] = [C]$

1. Decompose $[A]$ into $[L]$ and $[U]$
2. Solve $[L][Z] = [C]$ for $[Z]$
3. Solve $[U][X] = [Z]$ for $[X]$

Method: [A] Decompose to [L] and [U]

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

[*U*] is the same as the coefficient matrix at the end of the forward elimination step of Gaussian Elimination.

[*L*] is obtained using the *multipliers* that were used in the forward elimination process

Finding the [*U*] matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Step 1: $\dfrac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$

$$\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Finding the [U] Matrix

Matrix after Step 1: $$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Step 2: $$\frac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the [*L*] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

Finding the [L] Matrix

From the second step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

$$\ell_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Does [L][U] = [A]?

$$[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}= \quad ?$$

Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the [*L*] and [*U*] matrices

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example

Set $[L][Z] = [C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $[Z]$

$$z_1 = 10$$

$$2.56 z_1 + z_2 = 177.2$$

$$5.76 z_1 + 3.5 z_2 + z_3 = 279.2$$

Example

Complete the forward substitution to solve for $[Z]$

$$
\begin{aligned}
z_1 &= 106.8 \\
z_2 &= 177.2 - 2.56 z_1 \\
&= 177.2 - 2.56(106.8) \\
&= -96.2 \\
z_3 &= 279.2 - 5.76 z_1 - 3.5 z_2 \\
&= 279.2 - 5.76(106.8) - 3.5(-96.21) \\
&= 0.735
\end{aligned}
$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Set $[U][X] = [Z]$

Solve for $[X]$ The 3 equations become

$$\begin{aligned} 25a_1 + 5a_2 + a_3 &= 106.8 \\ -4.8a_2 - 1.56a_3 &= -96.21 \\ 0.7a_3 &= 0.735 \end{aligned}$$

Example

From the 3rd equation

$$0.7a_3 = 0.735$$

$$a_3 = \frac{0.735}{0.7}$$

$$a_3 = 1.050$$

Substituting in a_3 and using the second equation

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$a_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$a_2 = 19.70$$

Example

Substituting in a_3 and a_2 using the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$= \frac{106.8 - 5(19.70) - 1.050}{25}$$

$$= 0.2900$$

Hence the Solution Vector is:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$