Linear Algebra and Random Processes (CS6015)

DEPT. OF COMPUTER SCIENCE AND ENGINEERING
Indian Institute of Technology Madras

TUTORIAL 1
(Time allowed: FIFTY minutes)

NOTE: Attempt **ALL** questions. Total Marks : **25**

1. Give two examples of a 2×2 matrix A such that $A^2 = 0$ but $A \neq 0$. (2 marks)

Solution :
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ are two examples

2. If B is the inverse of A^2 , show that AB is the inverse of A. (2 marks)

Solution :
$A^2B = I$ can also be written as $A(AB) = I$. Therefore A^{-1} is AB.

3. Let $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$. For which X does there exist a scalar c such that $AX = cX$. (4 marks)

Solution:

$$\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

implies

$$5x = cx \tag{13}$$
$$x + 5y = cy \tag{14}$$
$$y + 5z = cz \tag{15}$$

Now if $c \neq 5$ then (13) implies $x = 0$, and then (14) implies $y = 0$, and then (15) implies $z = 0$. So it is true for $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ with $c = 0$.

If $c = 5$ then (14) implies $x = 0$ and (15) implies $y = 0$. So if $c = 5$ any such vector must be of the form $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$ and indeed any such vector works with $c = 5$.

So the final answer is any vector of the form $X = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$ and $z \in R$.

4. Show that $AB \neq BA$ in general using examples where $A, B \in R^{n \times n}$ (2 marks)

A $= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,

B $= \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

CONTINUED

$C = AB = \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix}$

$D = BA = \begin{bmatrix} 11 & 16 \\ 19 & 28 \end{bmatrix}$

Clearly $C \neq D$. Thereby $AB \neq BA$

5. Is the statement $tr(AB) = tr(A).tr(B)$ where $A, B \in R^{n \times n}$ correct? Validate your claim using an example. (2 marks)

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,

$B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

$C = AB = \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix}$

tr(A)tr(B) = 35 and tr(AB) = 39. Hence the statement is incorrect.

6. Apply Gaussian elimination and back substitution to solve

$$\begin{aligned} 2x - 3y &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

Circle the pivots and list the row operations. (3 marks)

Solution :

$$\begin{matrix} 2x - 3y \quad\quad = 3 & & 2x - 3y = 3 & & 2x - 3y = 3 & & x = 3 \\ 4x - 5y + \;\; z = 7 & \text{gives} & y + \;\; z = 1 & \text{and} & y + \;\; z = 1 & \text{and} & y = 1 \\ 2x - \;\; y - 3z = 5 & & 2y + 3z = 2 & & -5z = 0 & & z = 0 \end{matrix}$$

Here are steps $1, 2, 3$: Subtract $2 \times$ row 1 from row 2, subtract $1 \times$ row 1 from row 3, subtract $2 \times$ row 2 from row 3

7. Which number q makes the following system of linear equations, singular. Find the number t which gives infinitely many solutions, and the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t \end{aligned}$$

(3 marks)

Solution :

Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q + 4)z = t - 5$. If $q = -4$ the system is singular—no third pivot. Then if $t = 5$ the third equation is $0 = 0$ which allows infinitely many solutions. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

8. Find the triangular matrix E that reduces "Pascals matrix" to a smaller Pascal:

$$E\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&2&1\end{bmatrix}$$ (4 marks)

Hint: An elementary matrix E_{ij} subtracts ℓ times row j from row i. This E_{ij} includes $-\ell$ in row i, column j. An example is given below

$$E_{31}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\-\ell&0&1&0\\0&0&0&1\end{bmatrix}$$

Solution :

Construct and combine elementary matrices corresponding to operations:
$R4 \to R4 - R3$,
$R3 \to R3 - R2$, and
$R2 \to R2 - R1$

$$E=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}.$$

9. Prove that $||x||_\infty \leq ||x||_2 \leq ||x||_1$. Show from the Schwarz inequality ($|\langle u, v\rangle| \leq \|u\|\|v\|$. where u, v are vectors) that the ratios $||x||_2/||x||_\infty$ and $||x||_1/||x||_2$ are never larger than, $\sqrt{n}$. Which vector $(x_1, ..., x_n)$ gives ratios equal to $\sqrt{n}$? (3 marks)

Solution :

$x_1^2+\cdots+x_n^2$ is not smaller than $\max(x_i^2)$ and not larger than $(|x_1|+\cdots+|x_n|)^2 = \|\boldsymbol{x}\|_1^2$.

$x_1^2+\cdots+x_n^2 \leq n\,\max(x_i^2)$ so $\|\boldsymbol{x}\| \leq \sqrt{n}\|\boldsymbol{x}\|_\infty$. Choose $y_i = \operatorname{sign} x_i = \pm 1$ to get

$\|\boldsymbol{x}\|_1 = \boldsymbol{x}\cdot\boldsymbol{y} \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\| = \sqrt{n}\|\boldsymbol{x}\|$. The vector $\boldsymbol{x} = (1,\ldots,1)$ has $\|\boldsymbol{x}\|_1 = \sqrt{n}\,\|\boldsymbol{x}\|$.